Computational Robot Dynamics

Part 5: Forward Dynamics — The Articulated-Body Algorithm

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Introduction

We have just seen how to calculate forward dynamics using the recursive Newton-Euler algorithm (RNEA) and the composite-rigid-body algorithm (CRBA). The method consists of 3 steps:

1. Calculate $\boldsymbol{C}$ using the RNEA

2. Calculate $\boldsymbol{H}$ using the CRBA

3. Solve $\boldsymbol{\tau} = \boldsymbol{H}\ddot{\boldsymbol{q}} + \boldsymbol{C}$ for $\ddot{\boldsymbol{q}}$ using a factorization that exploits branch-induced sparsity.

The Articulated-Body Algorithm (ABA) is an algorithm for calculating $\ddot{\boldsymbol{q}}$ directly, without calculating either $\boldsymbol{H}$ or $\boldsymbol{C}$.

ABA versus RNEA+CRBA

Compared with calculating the dynamics using RNEA+CRBA, the ABA has the following advantages and disadvantages:

+ its computational complexity is $O(n)$
  (versus $O(nd^2)$ for RNEA+CRBA+factor&solve)

– it cannot handle kinematic loops.

Further Reading

We shall now proceed to derive the ABA for the special case of an unbranched kinematic chain with single-DoF joints, skipping some of the details. For a more complete description of the algorithm see

- Rigid Body Dynamics Algorithms
- Springer Handbook of Robotics
- Springer Encyclopedia of Robotics

and a complete implementation can be found in

- http://royfeatherstone.org/spatial/v2

The Basic Idea

The relationship between force and acceleration is linear.

Therefore, if we apply a force to any one body in a rigid-body system then the relationship between the applied force and the acceleration of the affected body can be expressed in the form

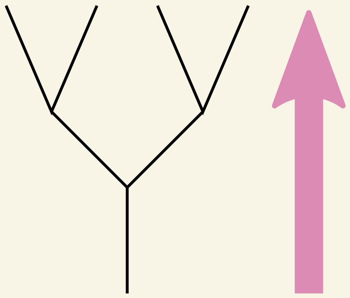
$$\boldsymbol{f} = \boldsymbol{I}^A \boldsymbol{a} + \boldsymbol{p}^A$$

where $\boldsymbol{f}$ is the applied force, $\boldsymbol{a}$ is the resulting acceleration, $\boldsymbol{I}^A$ expresses the inertia that the body appears to have, given the effects of other bodies in the system, and $\boldsymbol{p}^A$ is the bias force, which is the value of $\boldsymbol{f}$ to produce zero acceleration.

$\boldsymbol{I}^A$ is called an *articulated-body inertia*.

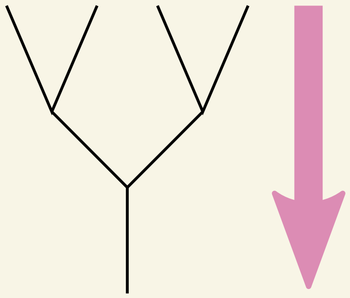
Overview

The ABA makes 3 passes through the tree as follows:



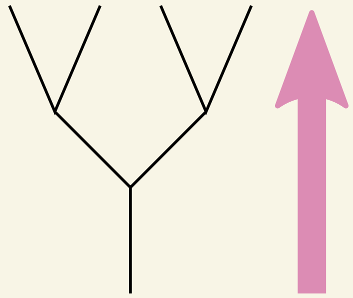
pass 1

calculate coordinate transforms and velocity terms



pass 2 (main pass)

calculate articulated-body inertias and bias forces



pass 3

calculate body and joint accelerations

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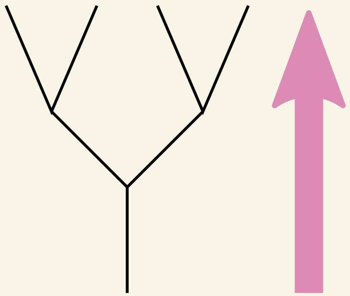
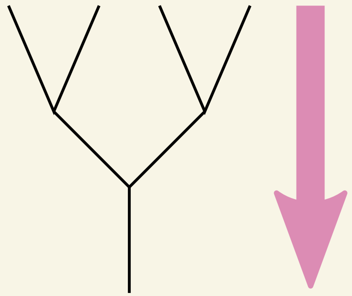


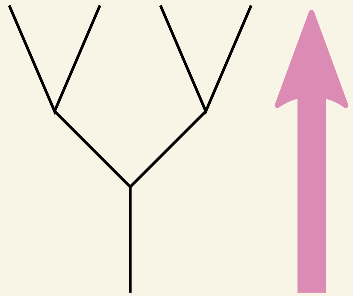
pass 1

calculate coordinate transforms and velocity terms



pass 2
(main pass)

calculate articulated-body inertias and bias forces



pass 3

calculate body and joint accelerations

We shall look at these two.

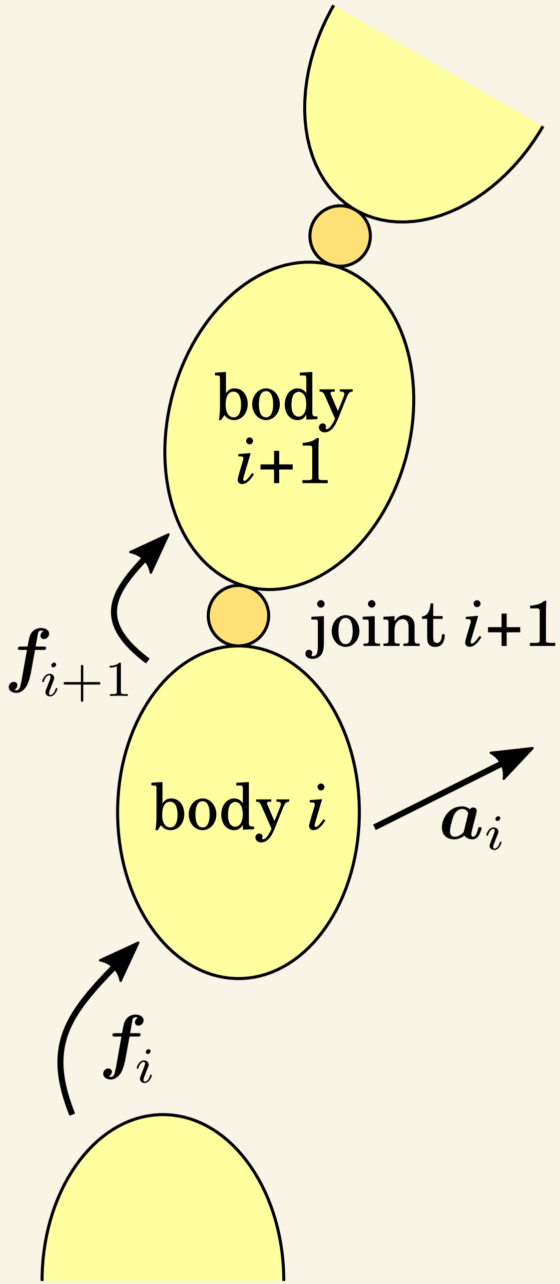
Main Pass

Suppose we break the chain at joint i, and consider the relationship between the force acting on body i and its resulting acceleration. The equation of motion is

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{p}_i$$

where

$$\boldsymbol{p}_i = \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$



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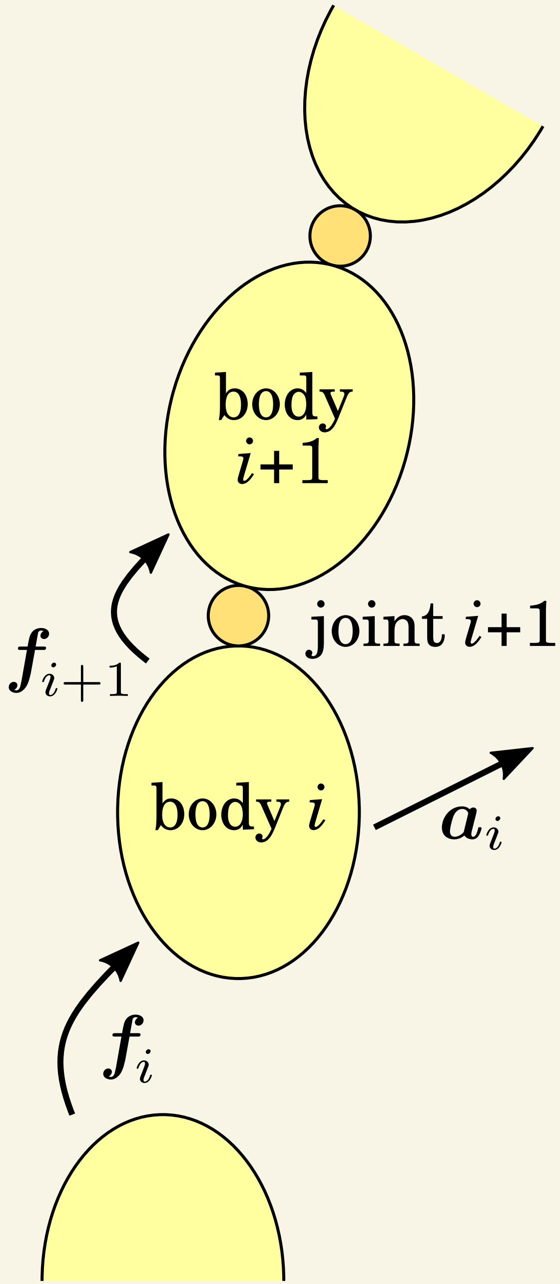
We seek an equation like this:

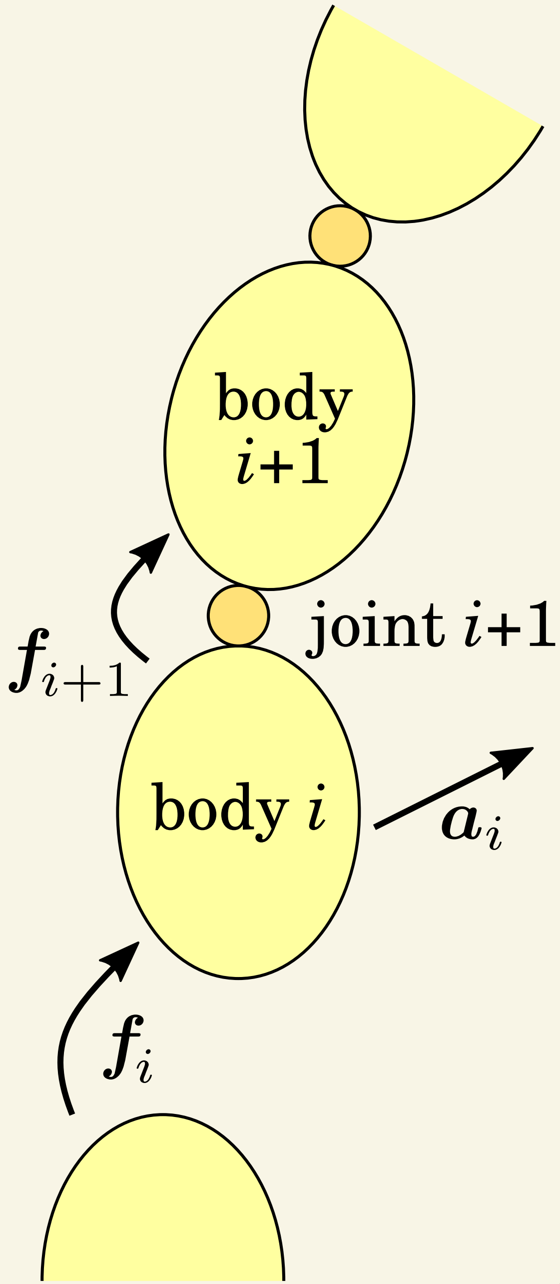
$$\boldsymbol{f}_i = \boldsymbol{I}_i^A \boldsymbol{a}_i + \boldsymbol{p}_i^A$$



Main Pass

To solve this problem we assume that $\boldsymbol{I}^A_{i+1}$ and $\boldsymbol{p}^A_{i+1}$ are already known, and look for a recurrence relation to calculate $\boldsymbol{I}^A_i$ and $\boldsymbol{p}^A_i$ from $\boldsymbol{I}^A_{i+1}$ and $\boldsymbol{p}^A_{i+1}$.

This works because $\boldsymbol{I}^A_N = \boldsymbol{I}_N$ and $\boldsymbol{p}^A_N = \boldsymbol{p}_N$.



Main Pass

Starting point:

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{p}_i$$

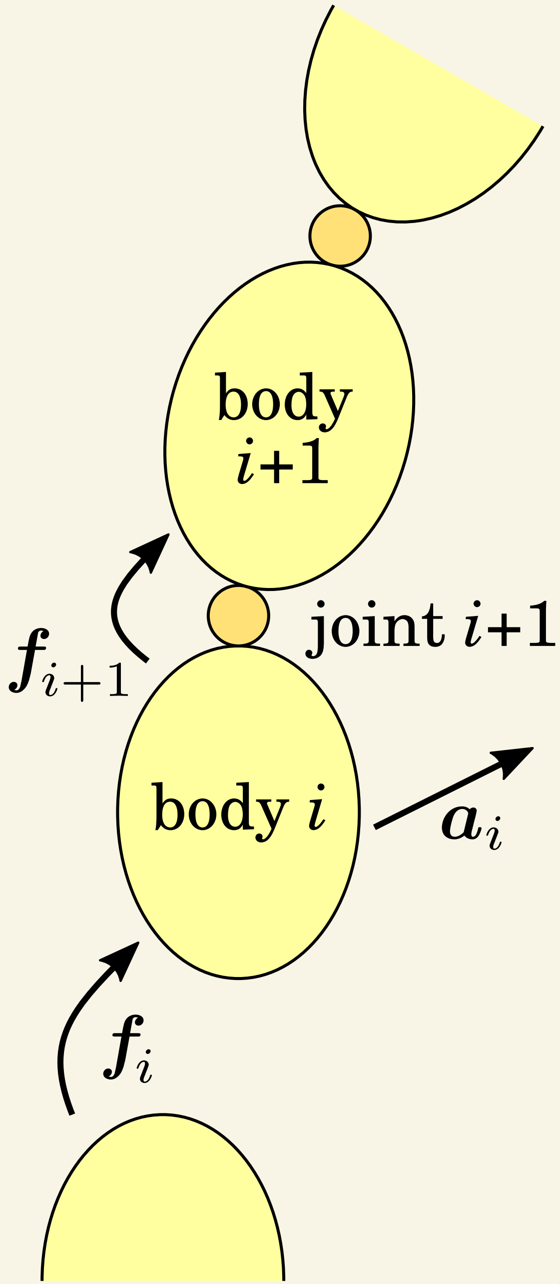
$$\boldsymbol{f}_{i+1} = \boldsymbol{I}^A_{i+1} \boldsymbol{a}_{i+1} + \boldsymbol{p}^A_{i+1}$$

$$\boldsymbol{a}_{i+1} = \boldsymbol{a}_i + \boldsymbol{c}_{i+1} + \boldsymbol{s}_{i+1}\, \ddot{q}_{i+1}$$

$$\boldsymbol{s}^{\mathrm{T}}_{i+1} \boldsymbol{f}_{i+1} = \tau_{i+1}$$

where $\boldsymbol{c}_{i+1} = \dot{\boldsymbol{s}}_{i+1}\, \dot{q}_{i+1}$ and both $\boldsymbol{p}_i$ and $\boldsymbol{c}_{i+1}$ are calculated in the first pass.

(Compare with sol6.pdf in your notes.)

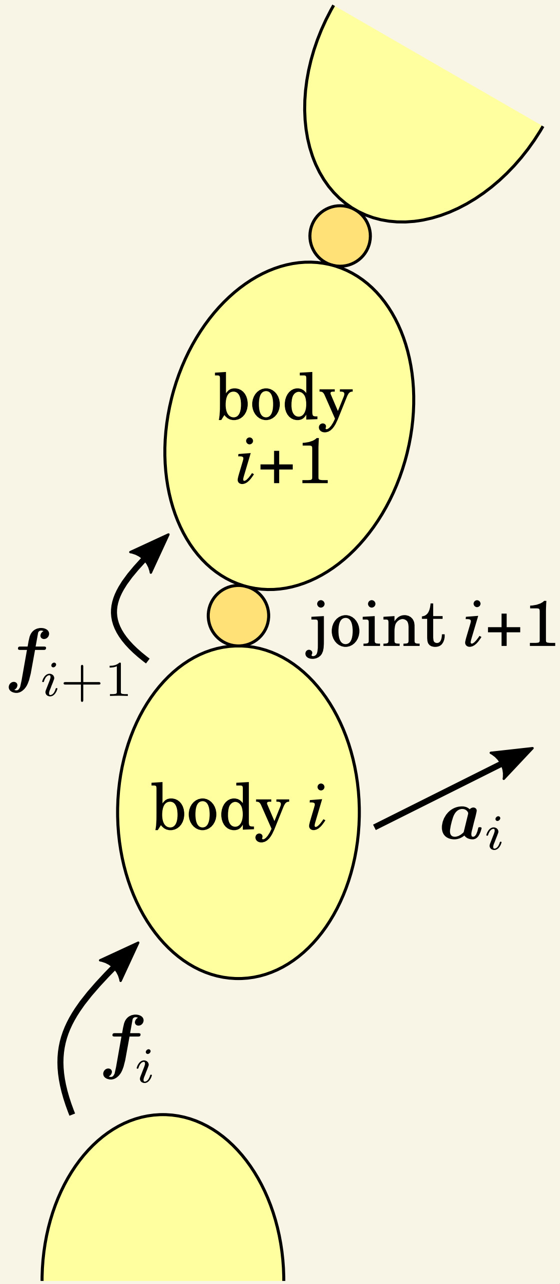


Main Pass

Step 1: expression for $\ddot{q}_{i+1}$

$$
\begin{aligned}
\tau_{i+1} &= \boldsymbol{s}_{i+1}^{\mathrm{T}} \boldsymbol{f}_{i+1} \\
&= \boldsymbol{s}_{i+1}^{\mathrm{T}} (\boldsymbol{I}_{i+1}^{A} \boldsymbol{a}_{i+1} + \boldsymbol{p}_{i+1}^{A}) \\
&= \boldsymbol{s}_{i+1}^{\mathrm{T}} (\boldsymbol{I}_{i+1}^{A} (\boldsymbol{a}_{i} + \boldsymbol{c}_{i+1} + \boldsymbol{s}_{i+1}\, \ddot{q}_{i+1}) \\
&\quad + \boldsymbol{p}_{i+1}^{A})
\end{aligned}
$$

$$
\ddot{q}_{i+1} = \frac{\tau_{i+1} - \boldsymbol{s}_{i+1}^{\mathrm{T}} (\boldsymbol{I}_{i+1}^{A} (\boldsymbol{a}_{i} + \boldsymbol{c}_{i+1}) + \boldsymbol{p}_{i+1}^{A})}{\boldsymbol{s}_{i+1}^{\mathrm{T}} \boldsymbol{I}_{i+1}^{A} \boldsymbol{s}_{i+1}}
$$



Main Pass

Introduce some common subexpressions:

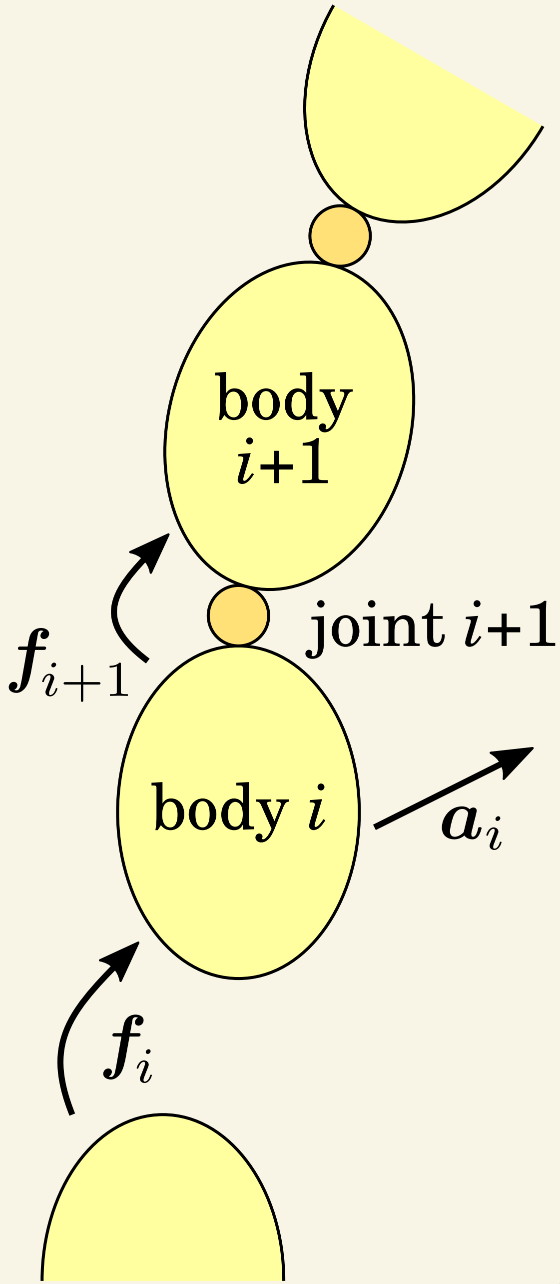
$$\boldsymbol{h}_{i+1} = \boldsymbol{I}^A_{i+1}\boldsymbol{s}_{i+1}$$

$$d_{i+1} = \boldsymbol{s}^{\mathrm{T}}_{i+1}\boldsymbol{h}_{i+1}$$

$$u_{i+1} = \tau_{i+1} - \boldsymbol{h}^{\mathrm{T}}_{i+1}\boldsymbol{c}_{i+1} - \boldsymbol{s}^{\mathrm{T}}_{i+1}\boldsymbol{p}^A_{i+1}$$

The expression for $\ddot{q}_{i+1}$ now simplifies to

$$\ddot{q}_{i+1} = \frac{u_{i+1} - \boldsymbol{h}^{\mathrm{T}}_{i+1}\boldsymbol{a}_i}{d_{i+1}}$$



Main Pass

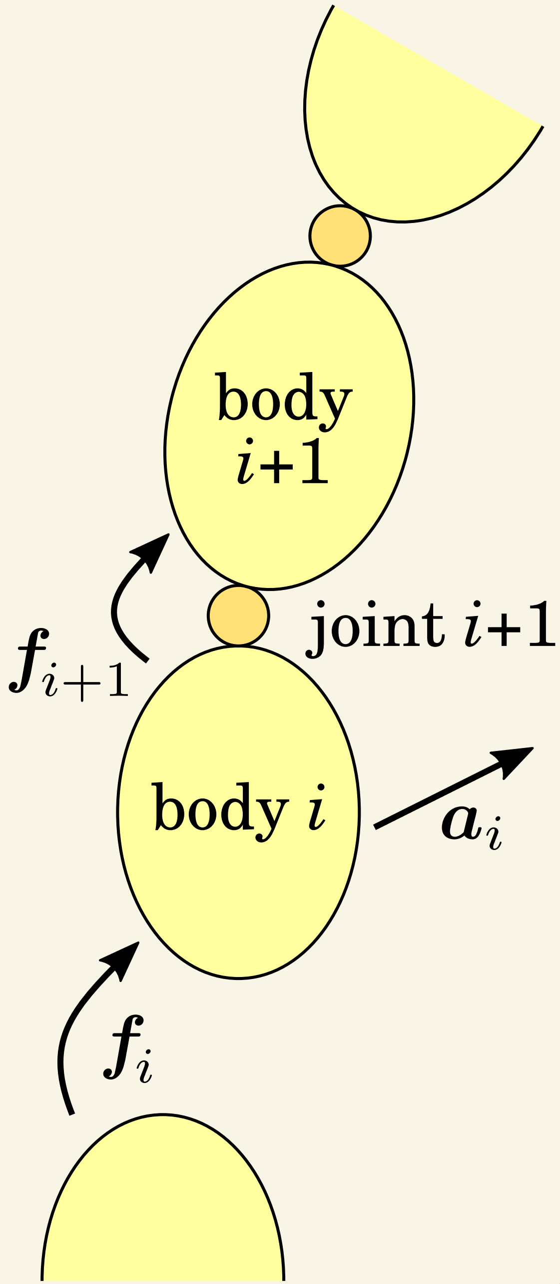
Step 2: expressions for $\boldsymbol{I}_i^A$ and $\boldsymbol{p}_i^A$

$$
\begin{aligned}
\boldsymbol{f}_i &= \boldsymbol{I}_i\boldsymbol{a}_i + \boldsymbol{p}_i + \boldsymbol{f}_{i+1} \\
&= \boldsymbol{I}_i\boldsymbol{a}_i + \boldsymbol{p}_i + \boldsymbol{I}_{i+1}^A\boldsymbol{a}_{i+1} + \boldsymbol{p}_{i+1}^A \\
&= \boldsymbol{I}_i\boldsymbol{a}_i + \boldsymbol{p}_i + \boldsymbol{I}_{i+1}^A\big(\boldsymbol{a}_i + \boldsymbol{c}_{i+1} + \\
&\quad \boldsymbol{s}_{i+1}(u_{i+1} - \boldsymbol{h}_{i+1}^{\mathrm{T}}\boldsymbol{a}_i)/d_{i+1}\big) + \boldsymbol{p}_{i+1}^A \\
&= \boldsymbol{I}_i^A\boldsymbol{a}_i + \boldsymbol{p}_i^A
\end{aligned}
$$

where

$$\boldsymbol{I}_i^A = \boldsymbol{I}_i + \boldsymbol{I}_{i+1}^A - \boldsymbol{h}_{i+1}\boldsymbol{h}_{i+1}^{\mathrm{T}}/d_{i+1}$$

$$\boldsymbol{p}_i^A = \boldsymbol{p}_i + \boldsymbol{p}_{i+1}^A + \boldsymbol{I}_{i+1}^A\boldsymbol{c}_{i+1} + \boldsymbol{h}_{i+1}\frac{u_{i+1}}{d_{i+1}}$$



Third Pass

Calculate body and joint accelerations

$$\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}}$$

$$\ddot{q}_i = \frac{u_i + \boldsymbol{h}_i^{\mathrm{T}} \boldsymbol{a}_{i-1}}{d_i}$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{i-1} + \boldsymbol{c}_i + \boldsymbol{s}_i \, \ddot{q}_i$$