An Introduction to Spatial (6D) Vectors

Answers Part 1

Topics: Vectors and Vector Fields
Instructor: Roy Featherstone, Italian Institute of Technology
Website: http://royfeatherstone.org/teaching

Answer A1

$\boldsymbol{a} = \sum_{j=1}^{n} a_j \boldsymbol{b}_j$, so $\boldsymbol{b}_i \cdot \boldsymbol{a} = \sum_{j=1}^{n} a_i (\boldsymbol{b}_i \cdot \boldsymbol{b}_j)$; but $\boldsymbol{b}_i \cdot \boldsymbol{b}_j = 0$ whenever $i \neq j$, and $\boldsymbol{b}_i \cdot \boldsymbol{b}_i = 1$, so $\boldsymbol{b}_i \cdot \boldsymbol{a} = a_i$.

Answer A2

second argument (columns); first argument (rows)

	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	$\boldsymbol{0}$	$\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}$	$-\boldsymbol{k}$	$\boldsymbol{0}$	$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}$	$\boldsymbol{0}$

Answer A3

Starting with the formula $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$ on Slide 11, we have

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} &= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \\
&= \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\mathbf{1}_{3\times 3}\boldsymbol{c} \\
&= \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{c} - (\boldsymbol{a} \cdot \boldsymbol{b})\mathbf{1}_{3\times 3}\boldsymbol{c} \\
&= (\boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} - (\boldsymbol{a} \cdot \boldsymbol{b})\mathbf{1}_{3\times 3})\boldsymbol{c}\,;
\end{aligned}$$

but this has to be true for all $\boldsymbol{c}$, which implies $\boldsymbol{a}\times\, \boldsymbol{b}\times = \boldsymbol{b}\boldsymbol{a}^{\mathrm{T}} - (\boldsymbol{a} \cdot \boldsymbol{b})\mathbf{1}_{3\times 3}$.

Answer A4

$\underline{\boldsymbol{a}} = [1.5, 0.5]^{\mathrm{T}}$, $\underline{\boldsymbol{c}} = [0.5, 1]^{\mathrm{T}}$, and $\underline{\boldsymbol{d}} = [0.5, 0.5]^{\mathrm{T}}$.

Answer A5

$\boldsymbol{e}_1 = [0.8, 0.48]^{\mathrm{T}}$ and $\boldsymbol{e}_2 = [0, 1]^{\mathrm{T}}$.
We will be encountering dual coordinates soon because they are the coordinates used by spatial (and planar) vectors. Dual coordinates arise in physics whenever a scalar invariant (like energy or power) is the result of a scalar product between two (physically) different types of vector, such as velocity and force. The use of dual coordinates preserves the invariance of the scalar product.

Answer B1

(a) A unit-magnitude rotation about the point $(0, 1)$.

(b) A rotation of magnitude 2 about the point $(-0.5, 0)$.

(c) A translation of magnitude $\sqrt{5}$ at an angle $\theta = \tan^{-1}(2)$ to the x axis.

Answer B2

One way to answer this question is to work out the velocity at P due to $\omega \boldsymbol{d}_O$ and velocity due to $r\omega \boldsymbol{d}_x$ and add them together. A rotation of magnitude ω about the origin produces a velocity at $P = (P_x, P_y)$ of $\omega(-P_y \boldsymbol{i} + P_x \boldsymbol{j})$. (This is a useful fact to memorize.) And the velocity due to $r\omega \boldsymbol{d}_x$ is $r\omega \boldsymbol{i}$. So the answer to the question is

$$\boldsymbol{V}(P) = \omega((r - P_y)\boldsymbol{i} + P_x \boldsymbol{j}) .$$

(Note: you can see from this answer that the velocity field is a rotation about the point $(0, r)$, because the velocity at this point is zero.)

Answer B3

We can deduce from their descriptions that $\boldsymbol{V}_1 = \boldsymbol{d}_O$ and $\boldsymbol{V}_2 = \boldsymbol{d}_O - \boldsymbol{d}_y$. So

$$\begin{aligned} \boldsymbol{V} &= \alpha \boldsymbol{d}_O + (1 - \alpha)(\boldsymbol{d}_O - \boldsymbol{d}_y) \\ &= \boldsymbol{d}_O - (1 - \alpha)\boldsymbol{d}_y . \end{aligned}$$

and $\boldsymbol{V}_1 - \boldsymbol{V}_2$ is simply $\boldsymbol{d}_y$. So we can say the following:

(a) It is a unit rotation about the point $(1 - \alpha, 0)$.

(b) It is a pure translation in the y direction.

In the case of item (a), observe that the rotation centre lies on the line passing through the two given rotation centres. In the case of item (b), we are summing two equal and opposite rotations $(\boldsymbol{V}_1 + (-\boldsymbol{V}_2))$, so the rotational component of the sum is zero, leaving only a translation perpendicular to the line joining the two rotation centres.

This result generalizes to two arbitrary rotations in the plane: given any two rotations, $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$, of any two magnitudes, about any two rotation centres, the vector field $\boldsymbol{V} = \alpha \boldsymbol{V}_1 + \beta \boldsymbol{V}_2$ is either a rotation about a point on the line passing through the two given rotation centres, or else a translation perpendicular to this line. The latter applies if the magnitude of the rotational component of $\boldsymbol{V}$ is zero.

In fact, this result also generalizes to 3D: given any two rotations about parallel axes in 3D space, any linear combination of these two rotations is either a rotation about an axis parallel to the given axes and lying in the same plane, or else a translation perpendicular to the plane containing the given axes.