An Introduction to Spatial (6D) Vectors and Their Use in Robot Dynamics

Part 3: Plücker Coordinates, Differentiation and Acceleration

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Coordinate Transforms — The General Rule

Let $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\} \subset U$ be a basis spanning a general vector space U, and let $B' = \{\boldsymbol{b}'_1, \boldsymbol{b}'_2, \ldots, \boldsymbol{b}'_n\} \subset U$ be a second basis also spanning U.

Let $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{u}}'$ be coordinate vectors representing the vector $\boldsymbol{u} \in U$ in the coordinate systems defined by B and B', respectively.

Let $\boldsymbol{X}$ be the coordinate transform (an $n \times n$ matrix) from B coordinates to B' coordinates, meaning that $\underline{\boldsymbol{u}}' = \boldsymbol{X}\underline{\boldsymbol{u}}$.

General Rule: the *columns* of $\boldsymbol{X}$ are the coordinates of the *original* basis vectors in the *new* coordinate system.

Coordinate Transforms — The General Rule

Proof: if both $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{u}}'$ represent $\boldsymbol{u}$ then

$$\sum_{i=1}^{n} u_i \boldsymbol{b}_i = \sum_{i=1}^{n} u_i' \boldsymbol{b}_i' = \boldsymbol{u}$$

and if the columns of $\boldsymbol{X}$ are the coordinates of $\boldsymbol{b}_i$ in B' then

$$\boldsymbol{b}_i = \sum_{j=1}^{n} X_{ji} \boldsymbol{b}_j'$$

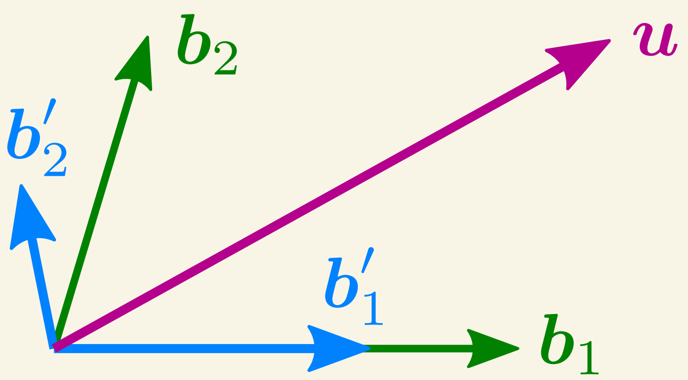
Therefore

$$\begin{aligned} \boldsymbol{u} &= \sum_{i=1}^{n} \sum_{j=1}^{n} u_i X_{ji} \boldsymbol{b}_j' \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} X_{ji} u_i \right) \boldsymbol{b}_j' \end{aligned}$$

which implies that $u_j' = \sum_{i=1}^{n} X_{ji} u_i$ hence $\underline{\boldsymbol{u}}' = \boldsymbol{X} \underline{\boldsymbol{u}}$

Coordinate Transforms — The General Rule

Example: (using Euclidean vectors)



$$\boldsymbol{u} = \boldsymbol{b}_1 + \boldsymbol{b}_2 \Rightarrow \underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{b}_1 = 1.5\,\boldsymbol{b}'_1$$

$$\boldsymbol{b}_2 = 0.5\,\boldsymbol{b}'_1 + 2\,\boldsymbol{b}'_2$$

$$\Downarrow$$

$$\boldsymbol{X} = \begin{bmatrix} 1.5 & 0.5 \\ 0 & 2 \end{bmatrix}$$

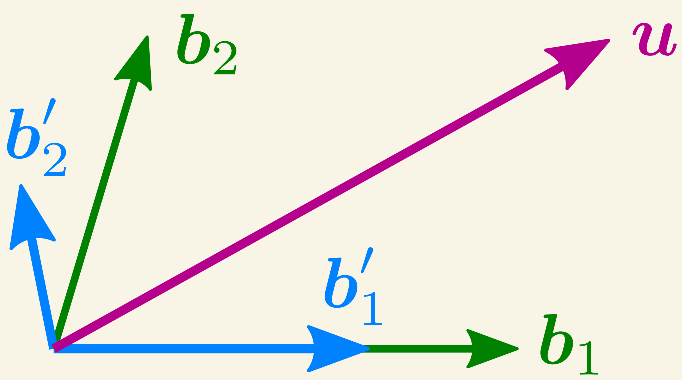
$$\underline{\boldsymbol{u}}' = \boldsymbol{X}\underline{\boldsymbol{u}} = \begin{bmatrix} 1.5 & 0.5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Memory aid: COIN

Columns **O**ld **I**n **N**ew

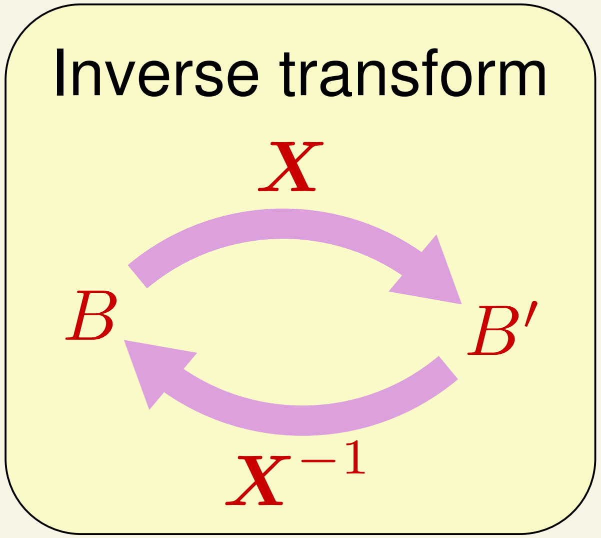
Coordinate Transforms — The General Rule

Example: (using Euclidean vectors)



$$\boldsymbol{u} = \boldsymbol{b}_1 + \boldsymbol{b}_2 \quad \Rightarrow \quad \underline{\boldsymbol{u}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{\boldsymbol{u}}' = \boldsymbol{X}\underline{\boldsymbol{u}} = \begin{bmatrix} 1.5 & 0.5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



Memory aid: COIN

Columns **O**ld **I**n **N**ew

Duality

It sometimes happens that vectors representing different physical quantities combine to form an invariant scalar. For example, *force* combines with *velocity* to form *power*.

When the physics has this form, it is sometimes useful to make the mathematical model have the same form; i.e.,

two vector spaces

plus

a scalar product defined *between* them

Duality

Let U and V be two vector spaces having the same dimension; and introduce a scalar product that takes one argument from each space.

- The scalar product makes each space *dual* to the other.
- The dual of a vector space U can be written U^*, so we have $V = U^*$ and $U = V^*$.
- As a notational convenience, if $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$ then we can write the scalar product either $\boldsymbol{u} \cdot \boldsymbol{v}$ or $\boldsymbol{v} \cdot \boldsymbol{u}$, both having the same meaning.

Dual Coordinate Systems

A *dual coordinate system* covers both spaces, and therefore uses *two bases*: one on each space.

Let $D = \{\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n\}$ and $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ be bases on U and V, respectively. To produce a valid dual coordinate system, the basis vectors must satisfy the following *reciprocity condition*:

$$\boldsymbol{d}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

If $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathrm{R}^n$ represent $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$ in a dual coordinate system then

$$\boldsymbol{u} \cdot \boldsymbol{v} = \underline{\boldsymbol{u}}^{\mathrm{T}} \underline{\boldsymbol{v}}$$

Dual Coordinate Transforms

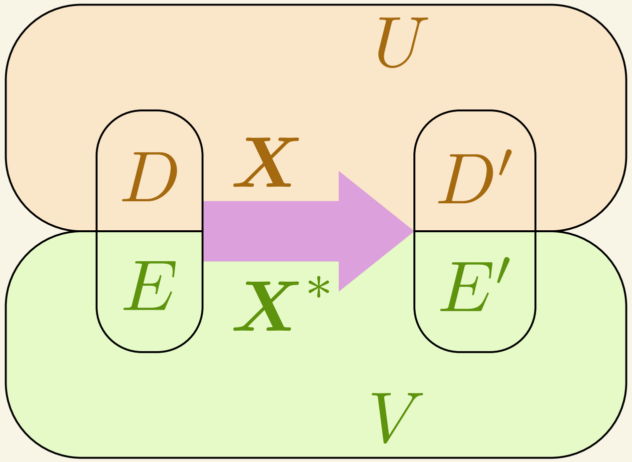
Let U be a general n-dimensional vector space; let D and D' be any two bases on U; and let $\boldsymbol{X}$ be the transform from D to D' coordinates.

Let $V = U^*$ be the dual of U; and let E and E' be the (unique) bases on V which are reciprocal to D and D', respectively, so that D and E together form a dual coordinate system on U and V, and D' and E' form another one.

The transform from E to E' coordinates, called $\boldsymbol{X}^*$, is then

$$\boldsymbol{X}^* = (\boldsymbol{X}^{-1})^{\mathrm{T}}$$

(can be written $\boldsymbol{X}^{-\mathrm{T}}$)



Dual Coordinate Transforms

Proof: Let $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in \mathrm{R}^n$ be coordinate vectors representing $\boldsymbol{u} \in U$ and $\boldsymbol{v} \in V$ in (D, E) coordinates, and let $\underline{\boldsymbol{u}}', \underline{\boldsymbol{v}}' \in \mathrm{R}^n$ represent $\boldsymbol{u}$ and $\boldsymbol{v}$ in (D', E') coordinates.

We require $\underline{\boldsymbol{u}}^{\mathrm{T}}\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}'^{\mathrm{T}}\underline{\boldsymbol{v}}'$ for every possible choice of $\boldsymbol{u}$ and $\boldsymbol{v}$, but $\underline{\boldsymbol{u}}'^{\mathrm{T}}\underline{\boldsymbol{v}}' = (\boldsymbol{X}\underline{\boldsymbol{u}})^{\mathrm{T}}(\boldsymbol{X}^*\underline{\boldsymbol{v}})$, so we require

$$\underline{\boldsymbol{u}}^{\mathrm{T}}\underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}^*\underline{\boldsymbol{v}}$$

for all $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$, and the only solution is

$$\boldsymbol{X}^{\mathrm{T}}\boldsymbol{X}^* = \boldsymbol{1}$$

which implies

$$\boldsymbol{X}^* = (\boldsymbol{X}^{\mathrm{T}})^{-1} = (\boldsymbol{X}^{-1})^{\mathrm{T}}$$

Dual Coordinate Transforms

Note:

- If U is Euclidean then it is possible to equate $U = U^*$. In other words, a Euclidean vector space can be self-dual.
- If $U = U^*$ then $D, E \subset U$.
- If $D, E \subset U$ and we choose $D = E$ then the dual coordinate system simplifies to a Cartesian coordinate system.
- This is why we don't need the concept of duality if we are working only with Euclidean vectors.

Mathematical Structure

spatial vectors inhabit *two* vector spaces:

M^6 — motion vectors

F^6 — force vectors

with a scalar product defined *between* them

$$\boldsymbol{m} \cdot \boldsymbol{f} = \mathit{work}$$

$$\text{“}\cdot\text{”} : \mathsf{M}^6 \times \mathsf{F}^6 \mapsto \mathsf{R}$$

Bases

A coordinate vector $\underline{\boldsymbol{m}}=[m_1,\ldots,m_6]^{\mathrm{T}}$ represents a motion vector $\boldsymbol{m}$ in a basis $\{\boldsymbol{d}_1,\ldots,\boldsymbol{d}_6\}$ on M^6 if

$$\boldsymbol{m} = \sum_{i=1}^{6} m_i \boldsymbol{d}_i$$

Likewise, a coordinate vector $\underline{\boldsymbol{f}}=[f_1,\ldots,f_6]^{\mathrm{T}}$ represents a force vector $\boldsymbol{f}$ in a basis $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_6\}$ on F^6 if

$$\boldsymbol{f} = \sum_{i=1}^{6} f_i \boldsymbol{e}_i$$

Bases

If $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_6\}$ is an arbitrary basis on M^6 then there exists a unique *reciprocal basis* $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_6\}$ on F^6 satisfying

$$\boldsymbol{d}_i \cdot \boldsymbol{e}_j = \begin{cases} 0 : & i \neq j \\ 1 : & i = j \end{cases}$$

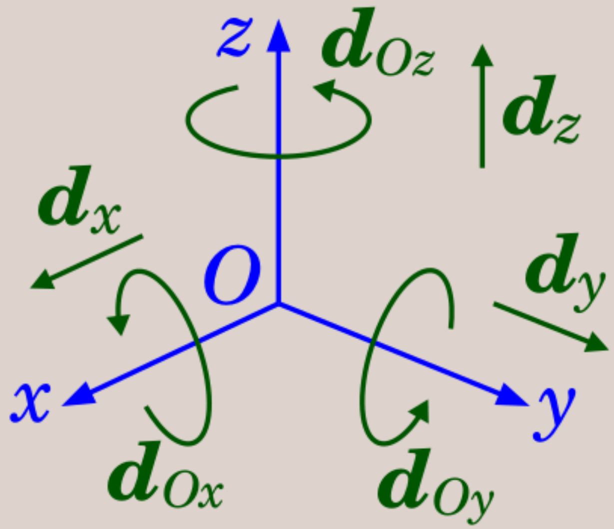
With these bases, the scalar product of two coordinate vectors is

$$\boldsymbol{m} \cdot \boldsymbol{f} = \underline{\boldsymbol{m}}^{\mathrm{T}} \underline{\boldsymbol{f}}$$

Plücker Coordinates

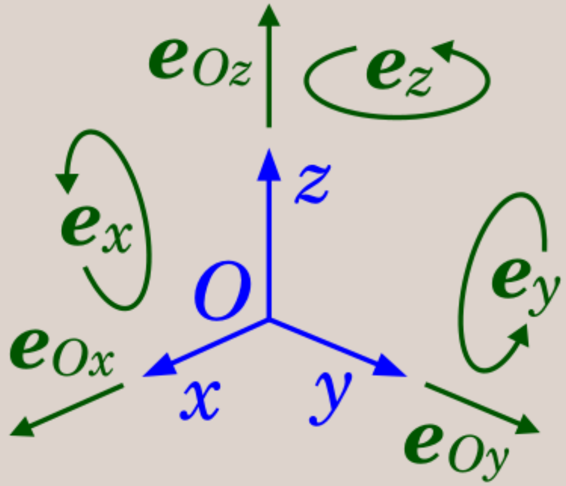
- Plücker coordinates are the standard coordinate system for spatial vectors
- a Plücker coordinate system is defined by the *position and orientation* of a *single* Cartesian frame
- a Plücker coordinate system has a total of *twelve* basis vectors, and covers both vector spaces (M^6 and F^6)

Define the following *Plücker basis* on M^6:



$\boldsymbol{d}_{Ox}$	unit angular motion about the line Ox
$\boldsymbol{d}_{Oy}$	unit angular motion about the line Oy
$\boldsymbol{d}_{Oz}$	unit angular motion about the line Oz
$\boldsymbol{d}_x$	unit linear motion in the x direction
$\boldsymbol{d}_y$	unit linear motion in the y direction
$\boldsymbol{d}_z$	unit linear motion in the z direction

Define the following
Plücker basis on F^6:



$\boldsymbol{e}_x$ unit couple in the x direction

$\boldsymbol{e}_y$ unit couple in the y direction

$\boldsymbol{e}_z$ unit couple in the z direction

$\boldsymbol{e}_{Ox}$ unit linear force along the line Ox

$\boldsymbol{e}_{Oy}$ unit linear force along the line Oy

$\boldsymbol{e}_{Oz}$ unit linear force along the line Oz

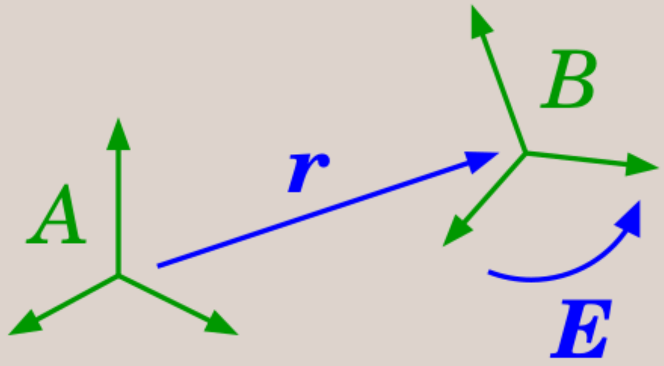
Plücker Coordinates

- the Plücker basis $\boldsymbol{e}_x, \boldsymbol{e}_y, \ldots, \boldsymbol{e}_{Oz}$ on F^6 is reciprocal to $\boldsymbol{d}_{Ox}, \boldsymbol{d}_{Oy}, \ldots, \boldsymbol{d}_z$ on M^6
- so the scalar product between a motion vector and a force vector can be expressed in Plücker coordinates as

$$\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{f}} = \underline{\hat{\boldsymbol{v}}}_O^{\mathrm{T}} \, \underline{\hat{\boldsymbol{f}}}_O$$

which is invariant with respect to the location of the coordinate frame

Coordinate Transforms



transform from A to B
for motion vectors:

$${}^{B}\boldsymbol{X}_{A} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \tilde{\boldsymbol{r}}^{\mathrm{T}} & \boldsymbol{1} \end{bmatrix} \quad \text{where} \quad \tilde{\boldsymbol{r}} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

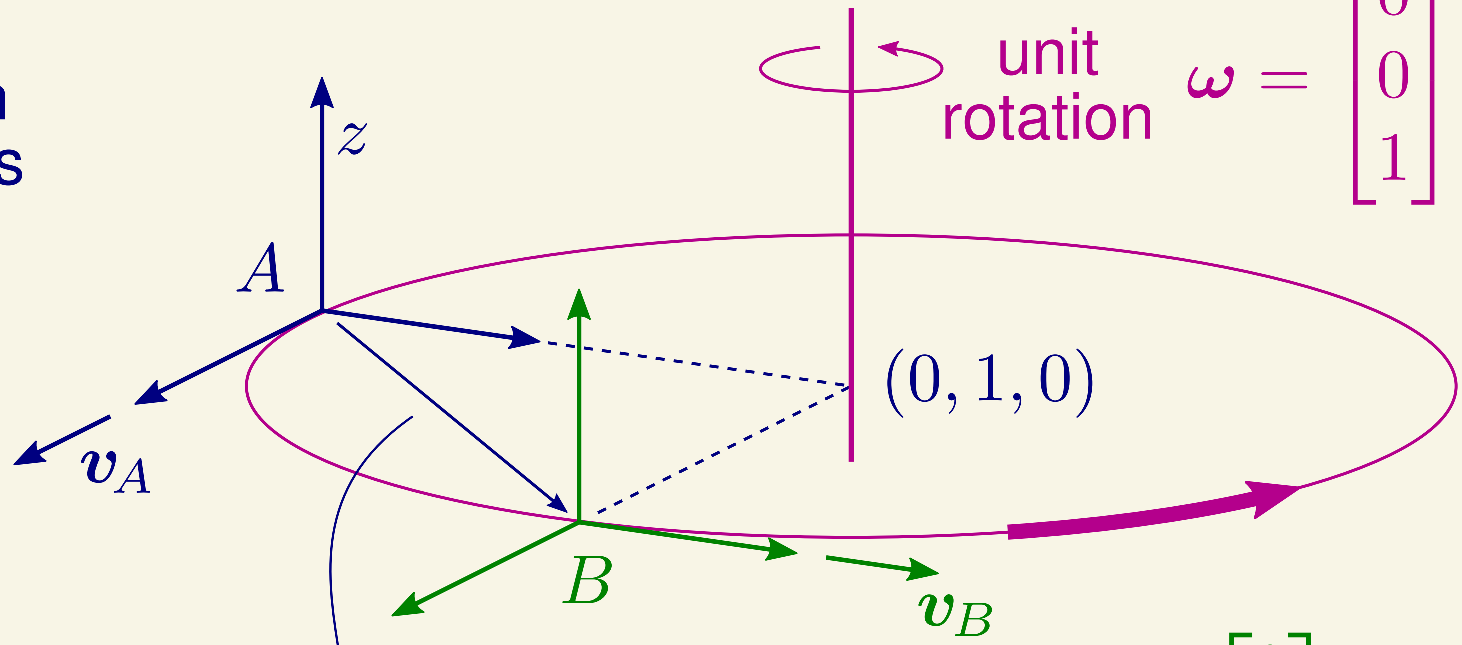
corresponding transform
for force vectors:

$${}^{B}\boldsymbol{X}_{A}^{*} = ({}^{B}\boldsymbol{X}_{A})^{-\mathrm{T}}$$

Coordinate Transform Example

expressed in
A coordinates

$$\hat{\boldsymbol{v}}_A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{r} = \overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

expressed in
B coordinates

$$\hat{\boldsymbol{v}}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Coordinate Transform Example

$$ {}^{B}\boldsymbol{X}_{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \boldsymbol{r}\times^{\mathrm{T}} & \mathbf{1} \end{bmatrix} $$

$$ \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\hat{\boldsymbol{v}}_B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\hat{\boldsymbol{v}}_A} $$

$$ \boldsymbol{r} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow \boldsymbol{r}\times^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} $$

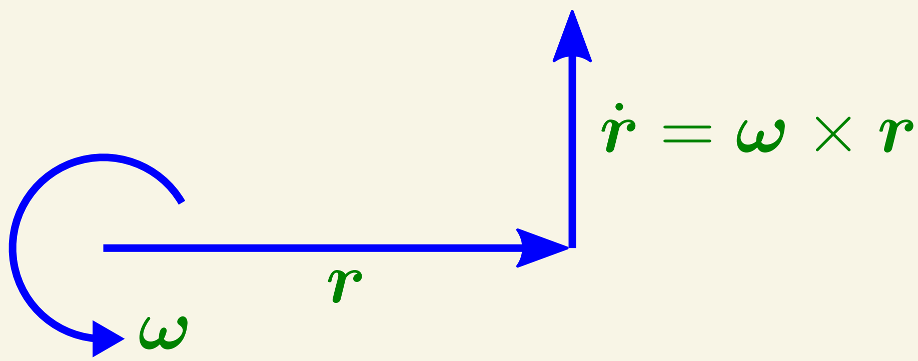
Now try question set A in part3Q.pdf

(answers in part3A.pdf)

Spatial Cross Products

If a Euclidean vector $\boldsymbol{r}$ is rotating with angular velocity $\boldsymbol{\omega}$ but not otherwise changing then its derivative is given by the formula

$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$$



We seek a spatial equivalent to this formula . . .

Spatial Cross Products

In particular, we seek a solution to the following problem:

Suppose that two spatial vectors, $\hat{\boldsymbol{m}} \in \mathsf{M}^6$ and $\hat{\boldsymbol{f}} \in \mathsf{F}^6$, are moving with a spatial velocity $\hat{\boldsymbol{v}}$ but not otherwise changing. Find formulae for the two 6x6 matrices $\hat{\boldsymbol{v}}\times$ and $\hat{\boldsymbol{v}}\times^*$ that satisfy

$$\dot{\hat{\boldsymbol{m}}} = \hat{\boldsymbol{v}}\times \hat{\boldsymbol{m}}$$

$$\dot{\hat{\boldsymbol{f}}} = \hat{\boldsymbol{v}}\times^* \hat{\boldsymbol{f}}$$

Spatial Cross Products

Skipping the details, the solution is

$$\hat{\boldsymbol{v}}\times = \begin{bmatrix} \boldsymbol{\omega}\times & \mathbf{0} \\ \boldsymbol{v}_O\times & \boldsymbol{\omega}\times \end{bmatrix}$$

where $\hat{\boldsymbol{v}} = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix}$

$$\hat{\boldsymbol{v}}\times^* = \begin{bmatrix} \boldsymbol{\omega}\times & \boldsymbol{v}_O\times \\ \mathbf{0} & \boldsymbol{\omega}\times \end{bmatrix}$$

A few properties:

$$\hat{\boldsymbol{v}}\times^* = -(\hat{\boldsymbol{v}}\times)^{\mathrm{T}}$$

$$\hat{\boldsymbol{v}}_1\times\hat{\boldsymbol{v}}_2 = -\hat{\boldsymbol{v}}_2\times\hat{\boldsymbol{v}}_1$$

(so $\hat{\boldsymbol{v}}\times\hat{\boldsymbol{v}} = \mathbf{0}$)

$$(\alpha\hat{\boldsymbol{v}})\times = \alpha(\hat{\boldsymbol{v}}\times)$$

$$(\hat{\boldsymbol{v}}_1 + \hat{\boldsymbol{v}}_2)\times = \hat{\boldsymbol{v}}_1\times + \hat{\boldsymbol{v}}_2\times$$

$$(\boldsymbol{X}\hat{\boldsymbol{v}})\times = \boldsymbol{X}\,\hat{\boldsymbol{v}}\times\boldsymbol{X}^{-1}$$

(Plücker transform)

Differentiation

- The derivative of a spatial vector is itself a spatial vector
- in general, $$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s} = \lim_{\delta t \to 0} \frac{\boldsymbol{s}(t+\delta t) - \boldsymbol{s}(t)}{\delta t}$$
- The derivative of a spatial vector that is fixed in a body moving with velocity $\boldsymbol{v}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s} = \begin{cases} \boldsymbol{v} \times \boldsymbol{s} & \text{if } \boldsymbol{s} \in \mathsf{M}^6 \\ \boldsymbol{v} \times^* \boldsymbol{s} & \text{if } \boldsymbol{s} \in \mathsf{F}^6 \end{cases}$$

Differentiation in Moving Coordinates

$$\underbrace{\left[\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s}\right]_O}_{\text{coordinate vector representing } \mathrm{d}\boldsymbol{s}/\mathrm{d}t} = \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s}_O}_{\text{componentwise derivative of coordinate vector } \boldsymbol{s}_O} + \underbrace{\boldsymbol{v}_O}_{\text{velocity of coordinate frame}} \times \boldsymbol{s}_O$$

or $\times^*$ if $\boldsymbol{s} \in \mathsf{F}^6$

Acceleration

. . . is the rate of change of velocity:

$$\hat{\boldsymbol{a}} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{v}} = \begin{bmatrix} \dot{\omega} \\ \dot{\boldsymbol{v}}_O \end{bmatrix}$$

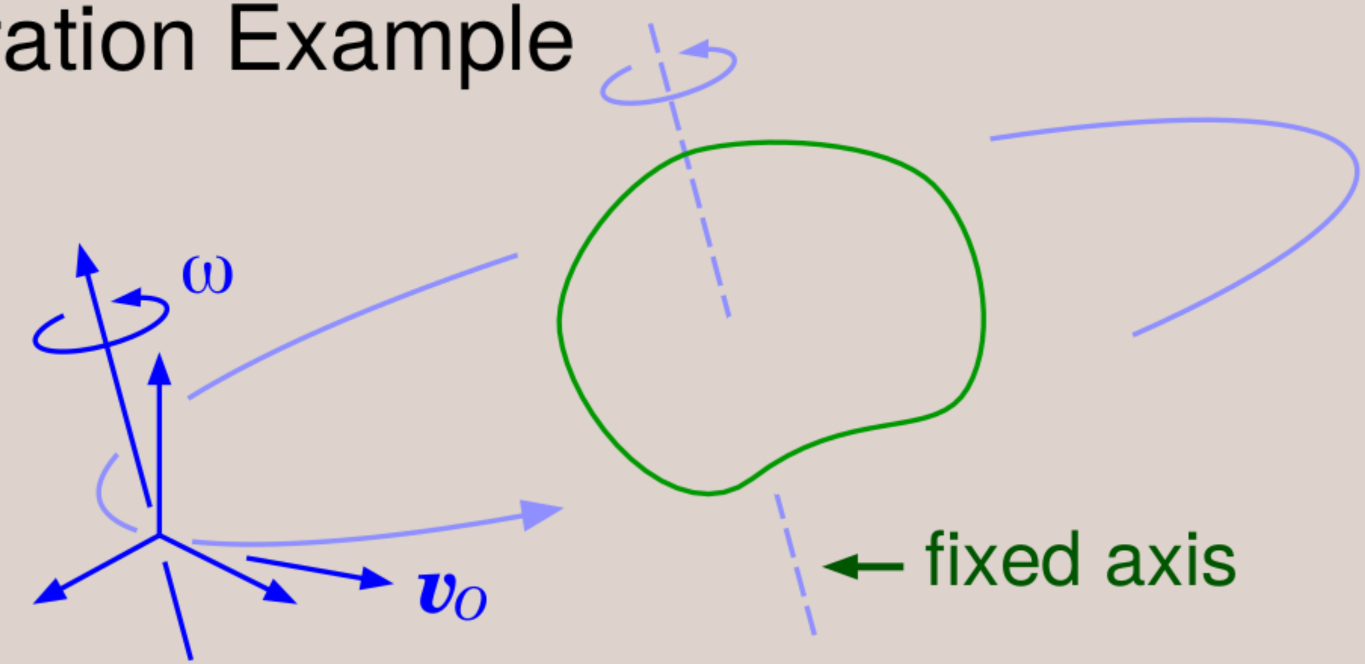
but this is *not* the linear acceleration of any point in the body!

Acceleration

- O is a fixed point in space,
- and $\boldsymbol{v}_O(t)$ is the velocity of the body−fixed point that coincides with O at time t,
- so $\boldsymbol{v}_O$ is the velocity at which body−fixed points are streaming through O.

- $\dot{\boldsymbol{v}}_O$ is therefore the rate of change of stream velocity

Acceleration Example



If a body rotates with constant angular velocity about a fixed axis, then its spatial velocity is constant and its spatial acceleration is zero; but each body–fixed point is following a circular path, and is therefore accelerating.

Acceleration Formula

Let $\boldsymbol{r}$ be the 3D vector giving the position of the body–fixed point that coincides with O at the current instant, measured relative to any fixed point in space

we then have $\hat{\boldsymbol{v}} = \begin{bmatrix} \omega \\ \boldsymbol{v}_O \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\boldsymbol{r}} \end{bmatrix}$

but $\hat{\boldsymbol{a}} = \begin{bmatrix} \dot{\omega} \\ \dot{\boldsymbol{v}}_O \end{bmatrix} = \begin{bmatrix} \dot{\omega} \\ \ddot{\boldsymbol{r}} - \omega \times \dot{\boldsymbol{r}} \end{bmatrix}$

Basic Properties of Acceleration

- Acceleration is the time−derivative of velocity
- Acceleration is a true vector, and has the same general algebraic properties as velocity
- Acceleration formulae are the derivatives of velocity formulae

If $\boldsymbol{v}_{\text{tot}} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ then $\boldsymbol{a}_{\text{tot}} = \boldsymbol{a}_1 + \boldsymbol{a}_2$

(Look, no Coriolis term!)

Now try question set B
in part3Q.pdf

(answers in part3A.pdf)