An Introduction to Spatial (6D) Vectors and Their Use in Robot Dynamics

Part 4: Dynamics

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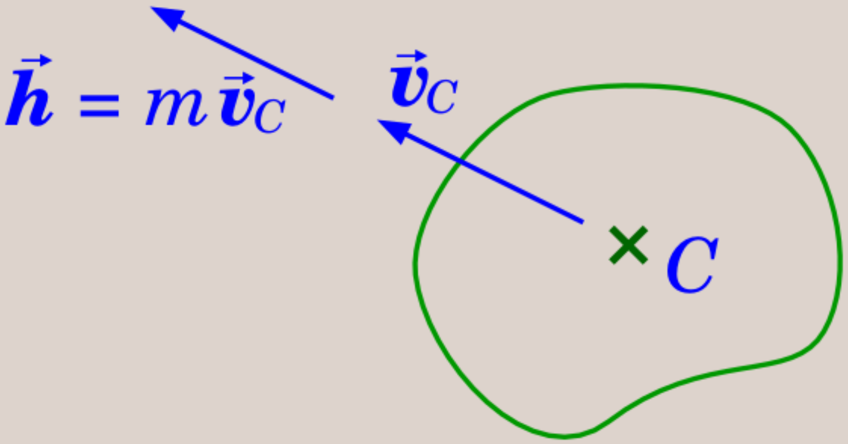
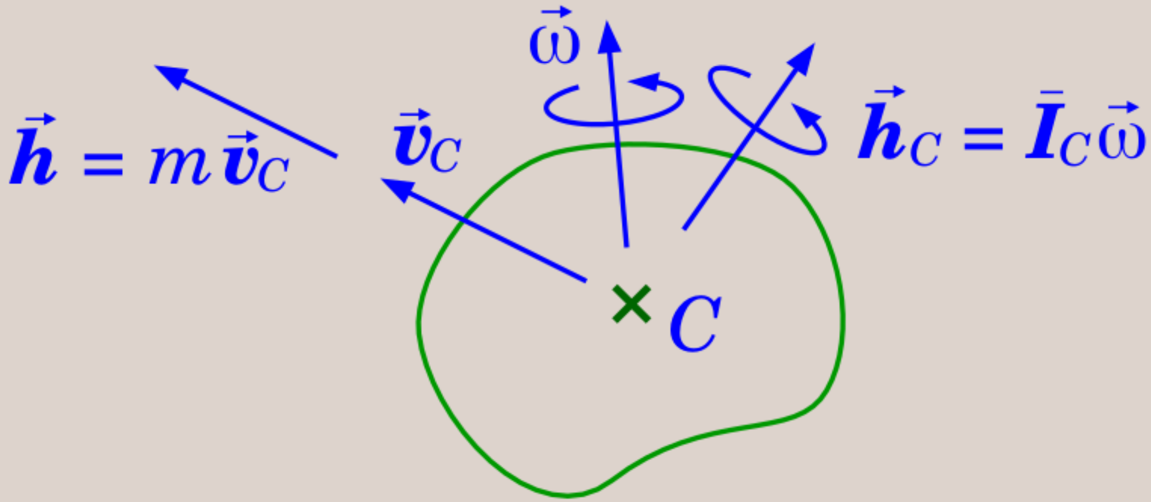
Momentum



The *linear momentum* of a rigid body is the product of its mass and the linear velocity of its centre of mass.

Linear momentum is a line vector.

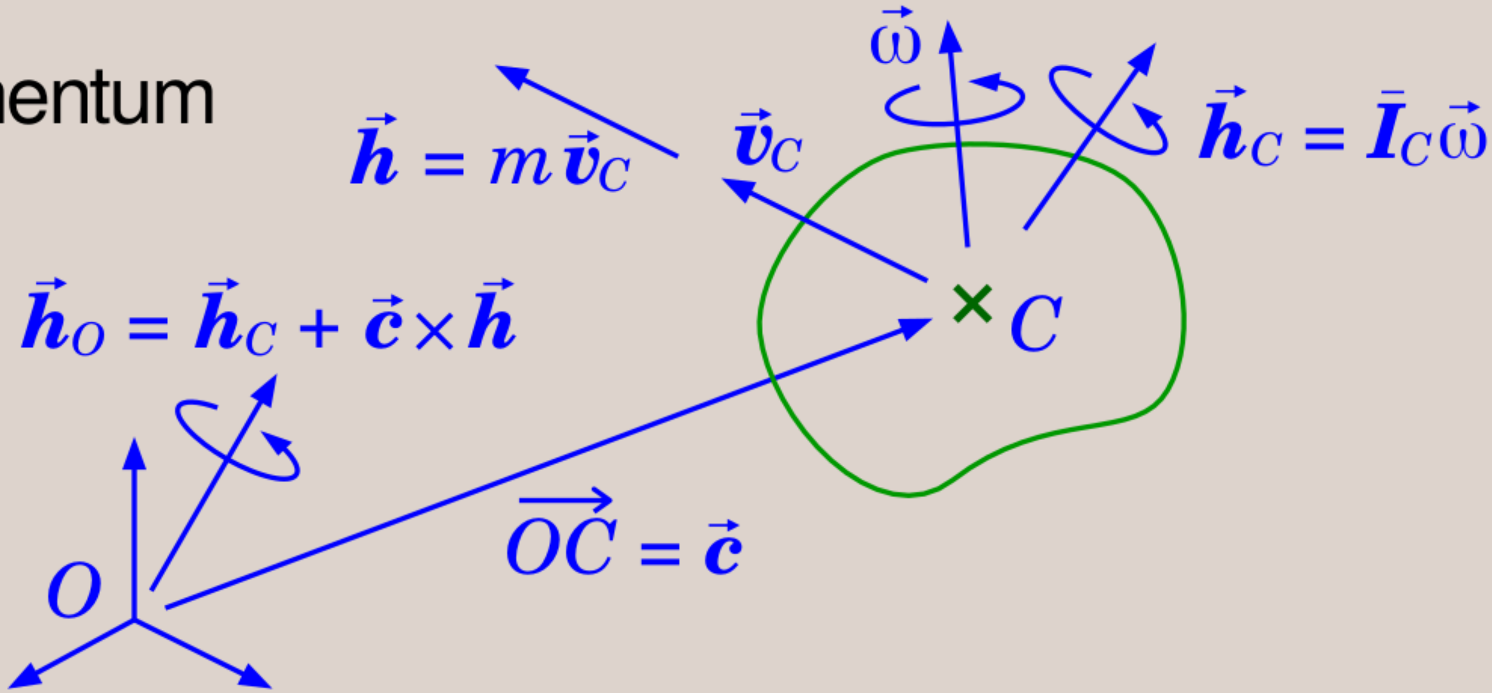
Momentum



The *intrinsic angular momentum* of a rigid body is the product of its angular velocity and its rotational inertia about the centre of mass.

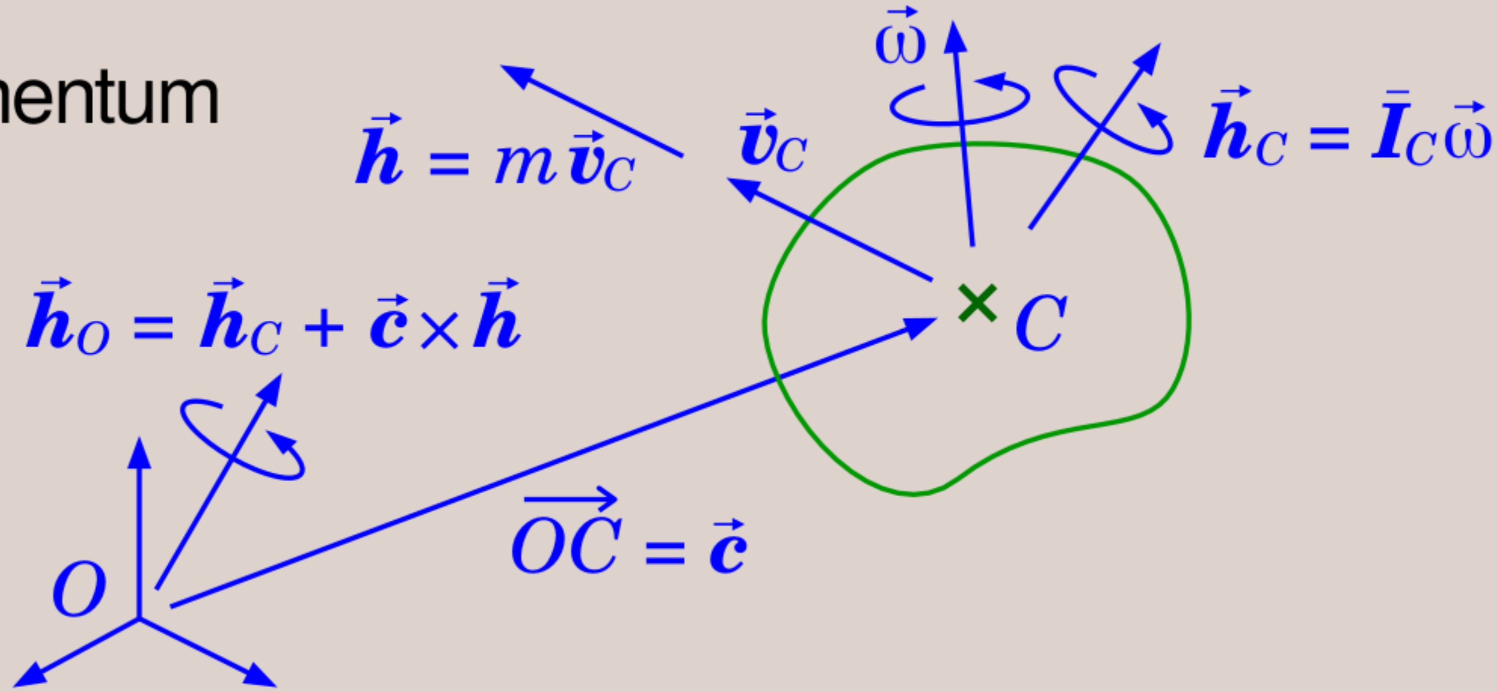
Angular momentum is a free vector.

Momentum



The *moment of momentum* of a rigid body about a given point O is the sum of its intrinsic angular momentum and the moment of its linear momentum about that point.

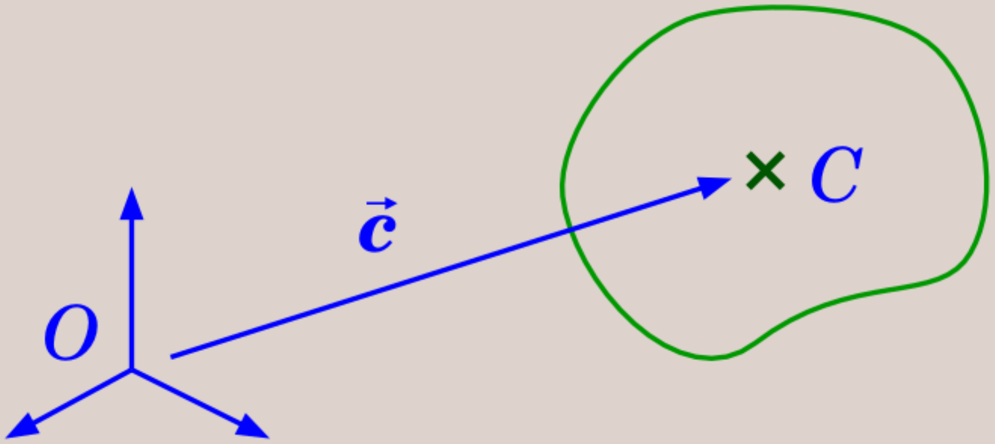
Momentum



The *spatial momentum* of a rigid body is a spatial force vector (a wrench) describing both its linear and angular momentum.

Plücker Coordinates: $\boldsymbol{h}_O = \begin{bmatrix} \vec{h}_O \\ \vec{h} \end{bmatrix}$

Inertia



mass: m

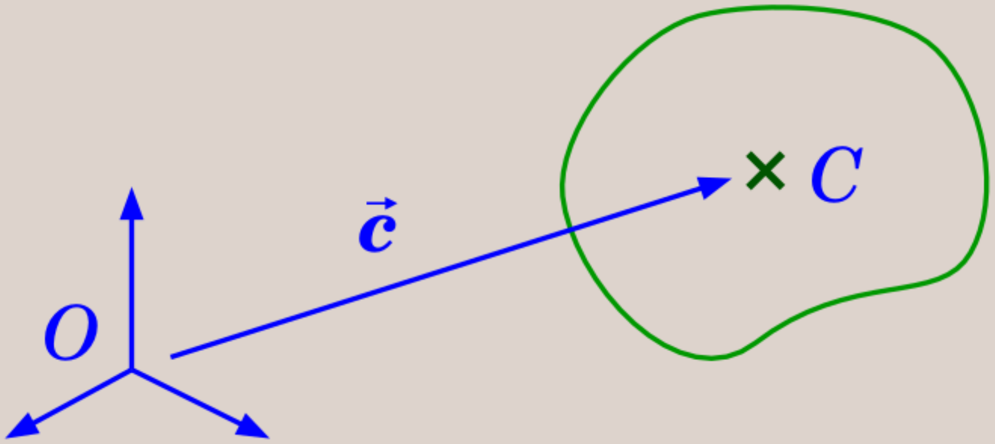
CoM: C

inertia at CoM: $\bar{\boldsymbol{I}}_C$

The *spatial inertia* of a rigid body is a function of its mass, centre of mass, and rotational inertia about its centre of mass.

Spatial inertia provides a complete description of a rigid body's inertia properties.

Inertia



mass: m

CoM: C

inertia at CoM: $\bar{\boldsymbol{I}}_C$

In Plücker coordinates, spatial rigid–body inertia is a 6×6 matrix:

$$\boldsymbol{I}_O = \begin{bmatrix} \bar{\boldsymbol{I}}_O & m\,\vec{\boldsymbol{c}}\times \\ m\,\vec{\boldsymbol{c}}\times^{\mathrm{T}} & m\,\mathbf{1} \end{bmatrix} \quad \text{where} \quad \bar{\boldsymbol{I}}_O = \bar{\boldsymbol{I}}_C - m\,\vec{\boldsymbol{c}}\times\vec{\boldsymbol{c}}\times$$

Properties of Inertia

- Tensor: Spatial inertia is the dyadic tensor that maps spatial velocity to momentum.
- Symmetry: $(\boldsymbol{I}\boldsymbol{v}_1)\cdot\boldsymbol{v}_2 = \boldsymbol{v}_1\cdot(\boldsymbol{I}\boldsymbol{v}_2)$
- Positive Definiteness: $\boldsymbol{v}\cdot\boldsymbol{I}\boldsymbol{v} > 0$ for all $\boldsymbol{v} \neq \boldsymbol{0}$
- Momentum: $\boldsymbol{h} = \boldsymbol{I}\boldsymbol{v}$
- Kinetic Energy: $T = \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{I}\boldsymbol{v}$

Properties of Inertia

- Time Derivative: If a rigid body has inertia $\boldsymbol{I}$ and velocity $\boldsymbol{v}$ then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{I} = \boldsymbol{v}\times^{*}\boldsymbol{I} - \boldsymbol{I}\,\boldsymbol{v}\times$$

- Coordinate Transformation Rule:

$$\boldsymbol{I}_B = {}^{B}\boldsymbol{X}_A^{*}\,\boldsymbol{I}_A\,{}^{A}\boldsymbol{X}_B = ({}^{A}\boldsymbol{X}_B)^{\mathrm{T}}\,\boldsymbol{I}_A\,{}^{A}\boldsymbol{X}_B$$

This is a congruence transform. It preserves symmetry and positive definiteness, but not eigenvalues or eigenvectors.

Properties of Inertia

- Composition: If two bodies having inertias of $\boldsymbol{I}_A$ and $\boldsymbol{I}_B$ are joined together then the inertia of the composite body is the sum:

$$\boldsymbol{I}_{\text{tot}} = \boldsymbol{I}_A + \boldsymbol{I}_B$$

- Number of Parameters: A rigid−body inertia is a function of 10 parameters. A general spatial inertia (e.g. for an articulated body) is a function of 21 parameters.

Equation of Motion

$$\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{I}\boldsymbol{v}) = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}$$

Force is the rate of change of momentum

$\boldsymbol{f}$ is the net force acting on a rigid body
$\boldsymbol{I}$ is the inertia of that body
$\boldsymbol{v}$ is the velocity of that body
$\boldsymbol{I}\boldsymbol{v}$ is the momentum of that body
$\boldsymbol{a}$ is the spatial acceleration of that body

Now try question set A in part4Q.pdf

(answers in part4A.pdf)

Motion Constraints

If a rigid body's motion is constrained, then its velocity is an element of a subspace $S \subset \mathsf{M}^6$, called the *motion freedom subspace*.

degree of (motion) freedom: $\dim(S)$

degree of constraint: $6 - \dim(S)$

S can vary with time

Motion Constraints

Motion constraints are caused by *constraint forces*, which have the following property:

> *A constraint force does no work against any motion allowed by the motion constraint*

(D'Alembert's principle of virtual work, and Jourdain's principle of virtual power)

Motion Constraints

Constraint forces are therefore elements of a constraint–force subspace, $T \subset \mathsf{F}^6$, which is the *orthogonal complement* (in the dual sense) of S. T is defined as follows:

$$T = \{ \boldsymbol{f} \mid \boldsymbol{f} \cdot \boldsymbol{v} = 0 \ \forall \boldsymbol{v} \in S \} = S^{\perp}$$

This subspace has the property

$$\dim(T) = 6 - \dim(S)$$

S and T provide equally good descriptions of the constraint.

Matrix Representation

- The subspace S can be represented by any $6 \times \dim(S)$ matrix $\boldsymbol{S}$ satisfying $\mathrm{range}(\boldsymbol{S}) = S$
- Likewise, the subspace T can be represented by any $6 \times \dim(T)$ matrix $\boldsymbol{T}$ satisfying $\mathrm{range}(\boldsymbol{T}) = T$

These matrices satisfy $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{T} = \boldsymbol{0}$

Note: the subspaces S and T are defined uniquely by the constraint, but the matrices representing them are not unique.

Constraint Equations

If $\boldsymbol{v}$ is any velocity allowed by the motion constraint then

$$\boldsymbol{v} \in S$$

$$\boldsymbol{v} = \boldsymbol{S}\,\alpha$$

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$$

where α is a $\dim(S) \times 1$ coordinate vector

If $\boldsymbol{f}$ is a constraint force then

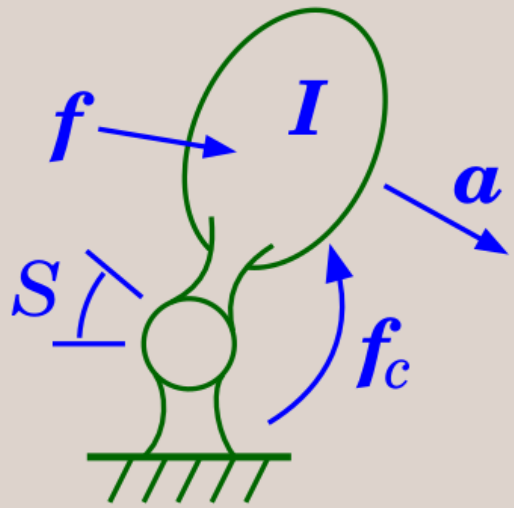
$$\boldsymbol{f} \in T$$

$$\boldsymbol{f} = \boldsymbol{T}\,\lambda$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f} = \mathbf{0}$$

where λ is a $\dim(T) \times 1$ coordinate vector

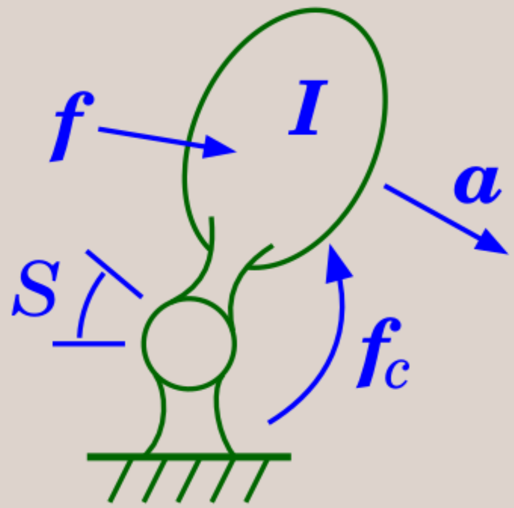
Constrained Motion Analysis



An Example

A force, $\boldsymbol{f}$, is applied to a body that is constrained to move in a subspace $S \subset \mathsf{M}^6$. The body has an inertia of $\boldsymbol{I}$, and is initially at rest. What is its acceleration, expressed as a function of $\boldsymbol{f}$?

Constrained Motion Analysis

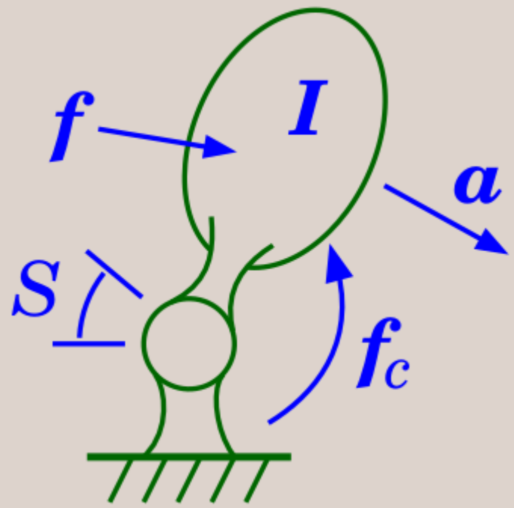


The total force acting on the body is $\boldsymbol{f}+\boldsymbol{f}_c$, where $\boldsymbol{f}_c$ is the constraint force induced by $\boldsymbol{f}$. The body's equation of motion is therefore

$$\boldsymbol{f}+\boldsymbol{f}_c=\boldsymbol{I}\boldsymbol{a}+\boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}$$

which simplifies to $\boldsymbol{f}+\boldsymbol{f}_c=\boldsymbol{I}\boldsymbol{a}$ because $\boldsymbol{v}=\boldsymbol{0}$.

Constrained Motion Analysis



This problem involves two unknowns: $\boldsymbol{f}_c$ and $\boldsymbol{a}$.

To solve the problem, we must first eliminate $\boldsymbol{f}_c$, and then express $\boldsymbol{a}$ in terms of $\boldsymbol{f}$.

There are two ways to proceed:

- using $\boldsymbol{S}$, or
- using $\boldsymbol{T}$

Let us try them both.

Relevant Equations:

$$\boldsymbol{v} = \boldsymbol{S}\alpha$$

$$\boldsymbol{a} = \boldsymbol{S}\dot{\alpha} + \dot{\boldsymbol{S}}\alpha$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}_c = \mathbf{0}$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v}$$

$\boldsymbol{v}=\mathbf{0}$ implies

$$\alpha = \mathbf{0}$$

$$\boldsymbol{a} = \boldsymbol{S}\dot{\alpha}$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a}$$

Relevant Equations:

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$$

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{a} + \dot{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$$

$$\boldsymbol{f}_c = \boldsymbol{T}\lambda$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v}$$

$\boldsymbol{v}=\mathbf{0}$ implies

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{a} = 0$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a}$$

Simplified Equations:

$$\boldsymbol{a} = \boldsymbol{S}\dot{\alpha}$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}_c = \boldsymbol{0}$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a}$$

Solution:

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{S}\dot{\alpha}$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\dot{\alpha}$$

$$\dot{\alpha} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}$$

$$\boldsymbol{a} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}$$

Simplified Equations:

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{f}_c = \boldsymbol{T}\lambda$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a}$$

Solution:

$$\boldsymbol{I}^{-1}(\boldsymbol{f} + \boldsymbol{f}_c) = \boldsymbol{a}$$

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}(\boldsymbol{f} + \boldsymbol{f}_c) = \boldsymbol{0}$$

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}(\boldsymbol{f} + \boldsymbol{T}\lambda) = \boldsymbol{0}$$

$$\vdots$$

Simplified Equations:

$$\boldsymbol{a} = \boldsymbol{S}\dot{\alpha}$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}_c = \boldsymbol{0}$$

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a}$$

Solution:

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{S}\dot{\alpha}$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\dot{\alpha}$$

$$\dot{\alpha} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}$$

$$\boldsymbol{a} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}$$

$$\vdots$$

Solution:

$$\boldsymbol{I}^{-1}(\boldsymbol{f} + \boldsymbol{f}_c) = \boldsymbol{a}$$

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}(\boldsymbol{f} + \boldsymbol{f}_c) = \boldsymbol{0}$$

$$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}(\boldsymbol{f} + \boldsymbol{T}\lambda) = \boldsymbol{0}$$

$$\lambda = -\boldsymbol{D}\,\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}\boldsymbol{f}$$

where $\boldsymbol{D} = (\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}\boldsymbol{T})^{-1}$

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{I}^{-1}(\boldsymbol{f} - \boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1}\boldsymbol{f}) \\ &= (\boldsymbol{I}^{-1} - \boldsymbol{I}^{-1}\boldsymbol{T}\boldsymbol{D}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{I}^{-1})\boldsymbol{f}\end{aligned}$$

Now try question set B in part4Q.pdf

(answers in part4A.pdf)

Robot Dynamics Calculations

Spatial vectors simplify the task of calculating robot dynamics. To illustrate their use, we shall look at one of the simplest, but most important dynamics algorithms: the *recursive Newton-Euler algorithm* (RNEA) for calculating the *inverse dynamics* of a robot mechanism.

The inverse dynamics problem: $\boldsymbol{\tau} = \mathrm{ID}(model, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$

given the position and velocity variables of every joint, and the desired joint accelerations, work out the joint forces required to produce those accelerations.

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \mathbf{0}$$

Initialization

$$\boldsymbol{a}_0 = \boxed{-\boldsymbol{a}_{\mathrm{g}}}$$

gravity is simulated by a fictitious base acceleration

$$\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

$$\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_{\mathrm{J}i}$$

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \mathbf{0}$$

$$\boldsymbol{a}_0 = -\boldsymbol{a}_\mathrm{g}$$

$$\boxed{\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i}$$

the velocity of each body is the sum of the velocity of its parent and the velocity of the joint connecting it to its parent

$$\boxed{\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i}$$

accelerations are defined likewise; this equation is just the derivative of the previous one

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

$$\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

$$\tau_i = \boldsymbol{s}_i^\mathrm{T} \boldsymbol{f}_{\mathrm{J}i}$$

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \boldsymbol{0}$$

$$\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boxed{\boldsymbol{s}_i \dot{q}_i}$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boxed{\boldsymbol{v}_i \times \boldsymbol{s}_i} \dot{q}_i$$

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

$$\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_{\mathrm{J}i}$$

joint velocity is the product of the joint axis vector, which defines the direction of motion, with the joint velocity variable, which defines the magnitude

$\dot{\boldsymbol{s}}_i = \boldsymbol{v}_i \times \boldsymbol{s}_i$ because $\boldsymbol{s}_i$ is fixed in body i, which is moving with velocity $\boldsymbol{v}_i$

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \boldsymbol{0}$$

$$\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i$$

$$\boxed{\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i}$$

the equation of motion calculates the forces required to produce the desired body accelerations

$$\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_{\mathrm{J}i}$$

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \mathbf{0}$$

$$\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

$$\boxed{\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}}$$

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_{\mathrm{J}i}$$

this equation calculates the spatial force transmitted across each joint

$\boldsymbol{f}_{\mathrm{J}i}$ is the force transmitted *from* body $\lambda(i)$ *to* body i through joint i, and $\boldsymbol{f}_{\mathrm{B}i}$ is the sum of all forces acting on body i, so

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{f}_{\mathrm{J}i} - \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

where $\mu(i)$ is the set of children of body i

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \boldsymbol{0}$$

$$\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

$$\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

$$\boxed{\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_{\mathrm{J}i}}$$

the joint force variable is the working component of the spatial force transmitted across the joint

Recursive Newton-Euler Algorithm

$$\boldsymbol{v}_0 = \mathbf{0}$$

$$\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}}$$

$$\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i$$

$$\boldsymbol{f}_{\mathrm{B}i} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

$$\boldsymbol{f}_{\mathrm{J}i} = \boldsymbol{f}_{\mathrm{B}i} + \textstyle\sum_{j \in \mu(i)} \boldsymbol{f}_{\mathrm{J}j}$$

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_{\mathrm{J}i}$$

$$
\begin{array}{l}
\boldsymbol{v}_0 = \mathbf{0} \\
\boldsymbol{a}_0 = -\boldsymbol{a}_{\mathrm{g}} \\
\textbf{for } i = 1 \textbf{ to } N \textbf{ do} \\
\quad [\boldsymbol{X}_{\mathrm{J}}, \boldsymbol{s}_i] = \mathrm{jcalc}(\mathrm{jtype}(i), q_i) \\
\quad {}^{i}\boldsymbol{X}_{\lambda(i)} = \boldsymbol{X}_{\mathrm{J}}\, \boldsymbol{X}_{\mathrm{T}}(i) \\
\quad \boldsymbol{v}_i = {}^{i}\boldsymbol{X}_{\lambda(i)} \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i \dot{q}_i \\
\quad \boldsymbol{a}_i = {}^{i}\boldsymbol{X}_{\lambda(i)} \boldsymbol{a}_{\lambda(i)} + \boldsymbol{s}_i \ddot{q}_i + \boldsymbol{v}_i \times \boldsymbol{s}_i \dot{q}_i \\
\quad \boldsymbol{f}_i = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i \\
\textbf{end} \\
\textbf{for } i = N \textbf{ to } 1 \textbf{ do} \\
\quad \tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_i \\
\quad \textbf{if } \lambda(i) \neq 0 \textbf{ then} \\
\qquad \boldsymbol{f}_{\lambda(i)} = \boldsymbol{f}_{\lambda(i)} + {}^{\lambda(i)}\boldsymbol{X}_i^* \boldsymbol{f}_i \\
\quad \textbf{end} \\
\textbf{end}
\end{array}
$$

Now try question set C in part4Q.pdf

(answers in part4A.pdf)