An Introduction to Spatial (6D) Vectors

Answers Part 2

Topics: Motion and Force
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Answer A1

(a) $\begin{bmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}$

Answer A2

(a) The answer can be obtained directly from the formula on slide 11. All you have to do is realize that $\boldsymbol{v}_P = \boldsymbol{V}(P)$ and rearrange this formula to put $\boldsymbol{v}_P$ on the left:

$$\boldsymbol{V}(P) = \boldsymbol{v}_O - \overrightarrow{OP} \times \boldsymbol{\omega}$$

(b) This part checks that you still remember the formula for the 3D Euclidean vector product. The answer is

$$\begin{aligned} \boldsymbol{V}(P) \;=\; & (v_{Ox} - P_y\omega_z + P_z\omega_y)\boldsymbol{i} + \\ & (v_{Oy} - P_z\omega_x + P_x\omega_z)\boldsymbol{j} + \\ & (v_{Oz} - P_x\omega_y + P_y\omega_x)\boldsymbol{k} \end{aligned}$$

Answer A3

The key to answering this question is to realize that $\boldsymbol{v}_O$ contains both a component in the direction of $\boldsymbol{\omega}$ and a component perpendicular to $\boldsymbol{\omega}$. The former contributes the linear motion along the screw axis, whereas the latter determines the position of this line in space. ($\boldsymbol{\omega}$ determines only its direction.) So the linear magnitude of the screwing motion is the magnitude of the component of $\boldsymbol{v}_O$ in the direction of $\boldsymbol{\omega}$.

(a) $\boldsymbol{v}_O \cdot \boldsymbol{\omega}/|\boldsymbol{\omega}|$

(b) $\boldsymbol{v}_O \cdot \boldsymbol{\omega}/\boldsymbol{\omega} \cdot \boldsymbol{\omega}$

Answer A4

We seek a point P such that $\boldsymbol{v}_P$ is parallel to $\boldsymbol{\omega}$, which implies that $\boldsymbol{\omega} \times \boldsymbol{v}_P = \mathbf{0}$. As $\boldsymbol{v}_P$ has the form $\boldsymbol{v}_P = \boldsymbol{v}_O + \boldsymbol{\omega} \times \overrightarrow{OP}$, we have

$$\boldsymbol{\omega} \times (\boldsymbol{v}_O + \boldsymbol{\omega} \times \overrightarrow{OP}) = \mathbf{0}\,,$$

which implies

$$\boldsymbol{\omega} \times \boldsymbol{\omega} \times \overrightarrow{OP} = -\boldsymbol{\omega} \times \boldsymbol{v}_O \,.$$

Applying the formula $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$ gives

$$(\boldsymbol{\omega} \cdot \overrightarrow{OP})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\overrightarrow{OP} = -\boldsymbol{\omega} \times \boldsymbol{v}_0 \,.$$

Now, there are infinitely many points P that lie on the instantaneous screw axis, including exactly one with the property that $\overrightarrow{OP}$ is perpendicular to the axis (i.e., perpendicular to $\boldsymbol{\omega}$). If we aim to find this one point then want a solution to the above equation in which $\boldsymbol{\omega} \cdot \overrightarrow{OP} = 0$. This gives us the following expression for P:

$$\overrightarrow{OP} = \frac{\boldsymbol{\omega} \times \boldsymbol{v}_O}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}} \,.$$

Answer B1

(a) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ \cos(\theta) \\ 1 + \sin(\theta) \end{bmatrix}$

Answer B2

(a) $\boldsymbol{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

(b) $\boldsymbol{v}_1 = \boldsymbol{s}_1 \, \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{s}_2 \, \dot{q}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \\ 0 \\ -\dot{q}_2 \\ 0 \end{bmatrix}$

(c) $\boldsymbol{J} = [\boldsymbol{s}_1 \; \boldsymbol{s}_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$

(d) $\boldsymbol{v}_P = \boldsymbol{v}_O - \overrightarrow{OP} \times \boldsymbol{\omega} = \begin{bmatrix} 0 \\ -\dot{q}_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\dot{q}_1 - \dot{q}_2 \\ \dot{q}_1 \\ 0 \end{bmatrix}$

(where O is the origin, and $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$ refer to $\hat{\boldsymbol{v}}_2$)

Answer B3

(a) Let $\hat{\boldsymbol{f}}$ be the spatial force equivalent to a 3D force of $\boldsymbol{f}$ acting on a line passing through P. The Plücker coordinates of $\hat{\boldsymbol{f}}$ are therefore

$$\hat{\boldsymbol{f}} = \begin{bmatrix} \overrightarrow{OP} \times \boldsymbol{f} \\ \boldsymbol{f} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} .$$

Let $\hat{\boldsymbol{f}}_1$ and $\hat{\boldsymbol{f}}_2$ be the forces transmitted from the base to B_1 through joint 1, and from B_1 to B_2 through joint 2, respectively. For static equilibrium, the net force on each body must be zero. The net force on B_1 is $\hat{\boldsymbol{f}}_1 - \hat{\boldsymbol{f}}_2$, and the net force on B_2 is $\hat{\boldsymbol{f}}_2 + \hat{\boldsymbol{f}}$; so the condition for static equilibrium is

$$\hat{\boldsymbol{f}}_1 = \hat{\boldsymbol{f}}_2 = -\hat{\boldsymbol{f}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} .$$

(b) $\tau_1 = \boldsymbol{s}_1^{\mathrm{T}} \hat{\boldsymbol{f}}_1 = -1$ and $\tau_2 = \boldsymbol{s}_2^{\mathrm{T}} \hat{\boldsymbol{f}}_2 = -1$.