# An Introduction to Spatial (6D) Vectors and Their Use in Robot Dynamics

## Part 1: Vectors and Vector Fields

Roy Featherstone

ISTITUTO ITALIANO
DI TECNOLOGIA
ADVANCED ROBOTICS

# Vectors and Vector Spaces

- A *vector* is, by definition, an element of a *vector space*.
- A *vector space* is, by definition, a mathematical structure consisting of an abelian group $G$, a field $K$, and a binary operator that maps $G \times K \to G$
- The elements of $G$ are called *vectors*, and the elements of $K$ are called *scalars*.
- Although the definition allows $K$ to be any field, we will use only the field of real numbers, R. So 'scalar' will always mean 'real number'.

# Vectors and Vector Spaces

- The group operator implements *vector addition*.
- So the group identity is the zero vector, $\mathbf{0}$, and the inverse of vector $\boldsymbol{a}$ is $-\boldsymbol{a}$.
- The binary operator implements *scalar multiplication*, and is required to be distributive over the group operator. So

$$\left.\begin{aligned}(\alpha+\beta)\boldsymbol{a} &= \alpha\boldsymbol{a}+\beta\boldsymbol{a}\\ \alpha(\boldsymbol{a}+\boldsymbol{b}) &= \alpha\boldsymbol{a}+\alpha\boldsymbol{b}\end{aligned}\right\} \quad \forall\, \alpha,\beta \in \mathsf{R},\ \boldsymbol{a},\boldsymbol{b} \in G$$

- Addition and scalar multiplication are the only two operations that must be defined on all vectors.

# Types of Vector

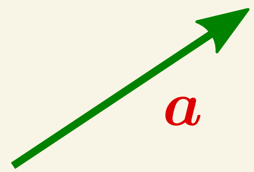
Most vectors have additional properties, which give rise to particular types of vector. For example:

| Type | Special Properties |
|---|---|
| coordinate vector | coordinates |
| Euclidean vector | magnitude and direction |
| spatial vector | two magnitudes and a directed line |

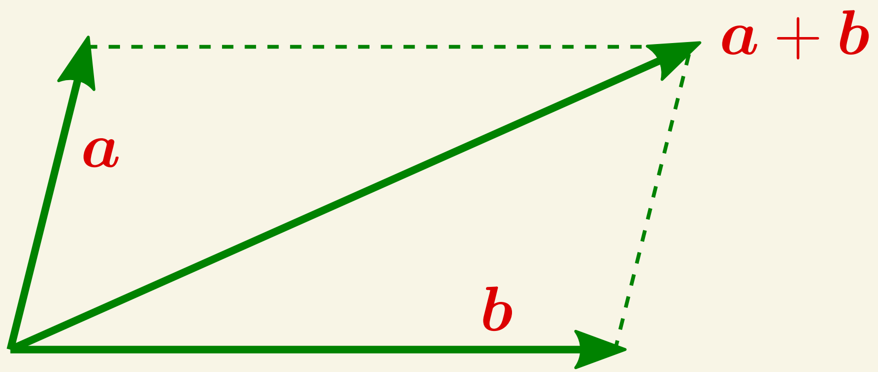
**Note:** Some textbooks define a vector to be an element of $\mathrm{R}^n$, but this is the definition of a coordinate vector, not a general vector.

# Euclidean Vectors (revision)

- A Euclidean vector can be represented graphically by an arrow. The length and direction of the arrow represent the magnitude and direction of the vector.

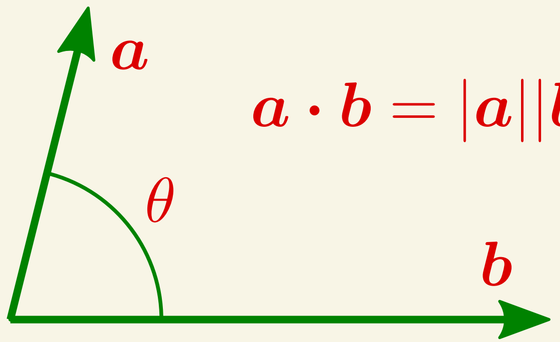



- Vector addition follows the parallelogram rule:



# Euclidean Vectors (revision)

- A *scalar product* (called 'dot product') is defined on Euclidean vectors as follows:




$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos(\theta)$$

where $|\boldsymbol{a}|$ denotes the magnitude of $\boldsymbol{a}$.

- So $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ if $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$ or $\boldsymbol{a}$ and $\boldsymbol{b}$ are at right angles (i.e., they are *orthogonal* to each other), and

- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$ (so $\boldsymbol{a} \cdot \boldsymbol{a} = 0$ if and only if $\boldsymbol{a} = \boldsymbol{0}$).

# Euclidean Vectors (revision)

- The scalar product is symmetrical ($\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$) and *bilinear*, which means that it is linear in each argument. So

  $$(\alpha \boldsymbol{a} + \beta \boldsymbol{b}) \cdot \boldsymbol{c} = \alpha \boldsymbol{a} \cdot \boldsymbol{c} + \beta \boldsymbol{b} \cdot \boldsymbol{c} \quad \forall\, \alpha, \beta \in \mathrm{R},\; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in G$$

  and similarly for $\boldsymbol{a} \cdot (\beta \boldsymbol{b} + \gamma \boldsymbol{c})$

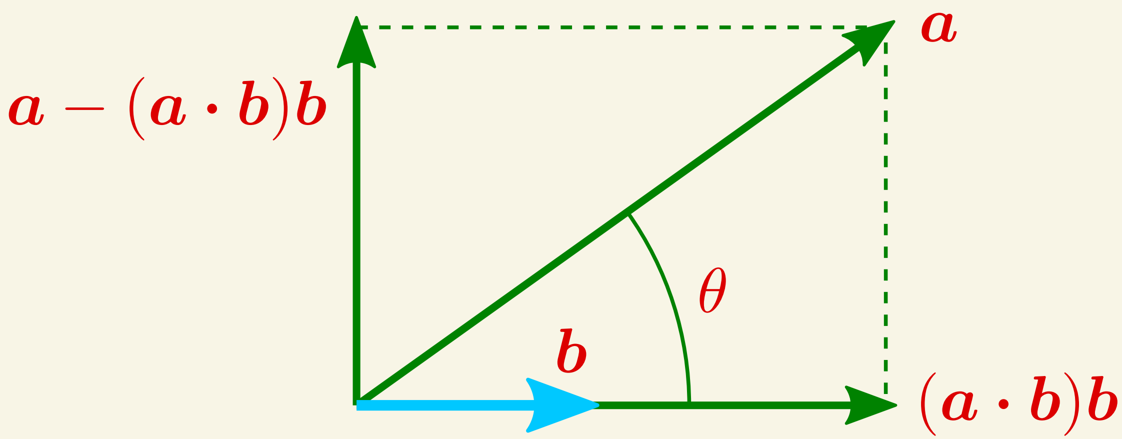
- Parentheses are not needed in expressions like $\alpha\beta \boldsymbol{a} \cdot \boldsymbol{b}$ because

  $$(\alpha\beta)(\boldsymbol{a} \cdot \boldsymbol{b}) = (\alpha\beta \boldsymbol{a}) \cdot \boldsymbol{b} = \alpha(\beta \boldsymbol{a}) \cdot \boldsymbol{b} = (\alpha \boldsymbol{a}) \cdot (\beta \boldsymbol{b})$$

  etc.

# Euclidean Vectors (revision)

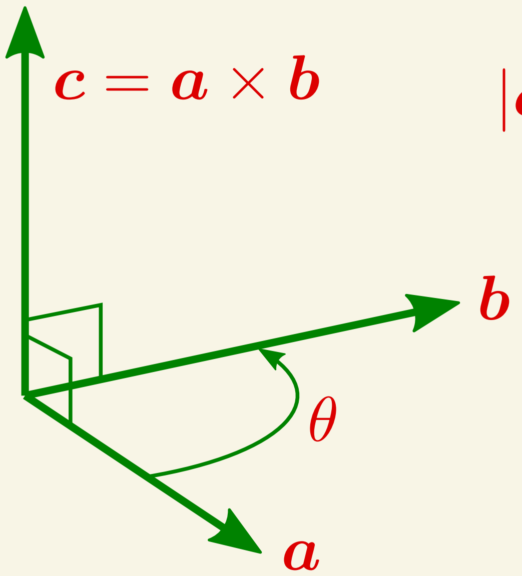
- If $\boldsymbol{b}$ is a *unit vector* (i.e., $|\boldsymbol{b}| = 1$) then $(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{b}$ is the *component* of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$, and $\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{b}$ is the component of $\boldsymbol{a}$ orthogonal to $\boldsymbol{b}$.




- The magnitudes of these two components are $|\boldsymbol{a}|\cos(\theta)$ and $|\boldsymbol{a}|\sin(\theta)$, respectively.

# Euclidean Vectors (revision)

- A *vector product* (called 'cross product') is defined on 3D Euclidean vectors as follows:




$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)$$

- $\boldsymbol{c}$ is at right angles to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and it points in the direction such that a positive rotation (in the right-hand rule sense) of $\boldsymbol{a}$ about $\boldsymbol{c}$ moves it closer to $\boldsymbol{b}$.

# Euclidean Vectors (revision)

- The vector product is bilinear, so
$$(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) \times \boldsymbol{c} = \alpha\boldsymbol{a} \times \boldsymbol{c} + \beta\boldsymbol{b} \times \boldsymbol{c}$$
etc.

- However, it is not symmetric ($\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$) and it is not associative:
$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \neq \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$$

- By convention, expressions like $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$ are interpreted as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ (i.e., the parentheses group to the right, which is the opposite of the convention for subtraction: $\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{c} = (\boldsymbol{a} - \boldsymbol{b}) - \boldsymbol{c}$).

# Euclidean Vectors (revision)

- A few more useful formulae:

$$|\boldsymbol{a}| + |\boldsymbol{b}| \geq |\boldsymbol{a} + \boldsymbol{b}| \qquad \text{(triangle inequality)}$$

$$\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} \qquad \text{(so-called 'triple product')}$$

$$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$$

$$\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}$$

- When using coordinate vectors (in Cartesian coordinates) we also have $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$, and $\boldsymbol{a} \times \boldsymbol{b}$ is the product of a 3x3 matrix $\boldsymbol{a}\times$ with the vector $\boldsymbol{b}$.

If $\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $\boldsymbol{a}\times = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$

# Euclidean Vectors (revision)

- Some properties of $\boldsymbol{a}\times$:

$$\boldsymbol{a}\times = -\boldsymbol{a}\times^{\mathrm{T}} \qquad \text{(skew symetry)}$$
$$(\lambda\boldsymbol{a})\times = \lambda(\boldsymbol{a}\times)$$
$$(\boldsymbol{a}+\boldsymbol{b})\times = \boldsymbol{a}\times + \boldsymbol{b}\times$$
$$(\boldsymbol{E}\boldsymbol{a})\times = \boldsymbol{E}\,\boldsymbol{a}\times\,\boldsymbol{E}^{-1} \qquad \text{(where } \boldsymbol{E} \text{ is 3x3 orthogonal)}$$
$$\boldsymbol{a}\times\boldsymbol{b}\times = \boldsymbol{b}\,\boldsymbol{a}^{\mathrm{T}} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{1}_{3\times 3}$$
$$(\boldsymbol{a}\times\boldsymbol{b})\times = \boldsymbol{a}\times\boldsymbol{b}\times - \boldsymbol{b}\times\boldsymbol{a}\times = \boldsymbol{b}\,\boldsymbol{a}^{\mathrm{T}} - \boldsymbol{a}\,\boldsymbol{b}^{\mathrm{T}}$$

Note:

- alternative notations exist for $\boldsymbol{a}\times$, such as $\tilde{\boldsymbol{a}}$
- the cross in $\boldsymbol{a}\times$ binds tightly to the 'a' so that expressions like $\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{c}$ mean $(\boldsymbol{a}\times)(\boldsymbol{b}\times)\boldsymbol{c}$, which is the same thing as $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ on slide 10.

# Euclidean Vectors (revision)

A worked example: prove $\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$

solution:
$$
\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c} &= (\boldsymbol{a} \times \boldsymbol{b})^{\mathrm{T}} \boldsymbol{c} \\
&= (-\boldsymbol{b} \times \boldsymbol{a})^{\mathrm{T}} \boldsymbol{c} \\
&= -\boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}\times^{\mathrm{T}} \boldsymbol{c} \\
&= \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}\times \boldsymbol{c} \\
&= \boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}
\end{aligned}
$$

Tip: It makes no difference whether you think of $\boldsymbol{a} \times \boldsymbol{b}$ as the cross product of $\boldsymbol{a}$ and $\boldsymbol{b}$ or as the product of $\boldsymbol{a}\times$ with $\boldsymbol{b}$. The two are equivalent.

# Bases and Coordinates (revision)

Let $U$ be an $n$-dimensional vector space, and let $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m\} \subset U$ be a set of $m$ vectors in $U$, $m \leq n$.

- A set of vectors $\boldsymbol{b}_i$ is said to be *linearly independent* if the only solution to the equation $\sum_{i=1}^{m} \beta_i \boldsymbol{b}_i = \boldsymbol{0}$ is $\beta_i = 0$.
- If the vectors in $B$ are linearly independent then $B$ spans an $m$-dimensional subspace of $U$ (denoted $\mathrm{span}(B)$) and forms a *basis* on it.
- In this case, any vector $\boldsymbol{a} \in \mathrm{span}(B)$ can be expressed uniquely as a linear combination of the elements of $B$:

  $$\boldsymbol{a} = \sum_{i=1}^{m} a_i \boldsymbol{b}_i$$

  and the scalars $a_i$ are the *coordinates* of $\boldsymbol{a}$ in the basis $B$.

# Bases and Coordinates (revision)

So the basis $B$ defines a *coordinate system* on $\mathrm{span}(B)$, which we will call $B$ coordinates. Any vector $\boldsymbol{a} \in \mathrm{span}(B)$ can be represented in $B$ coordinates by a coordinate vector $\underline{\boldsymbol{a}} = [a_1\ a_2\ \cdots\ a_m]^{\mathrm{T}} \in \mathsf{R}^m$

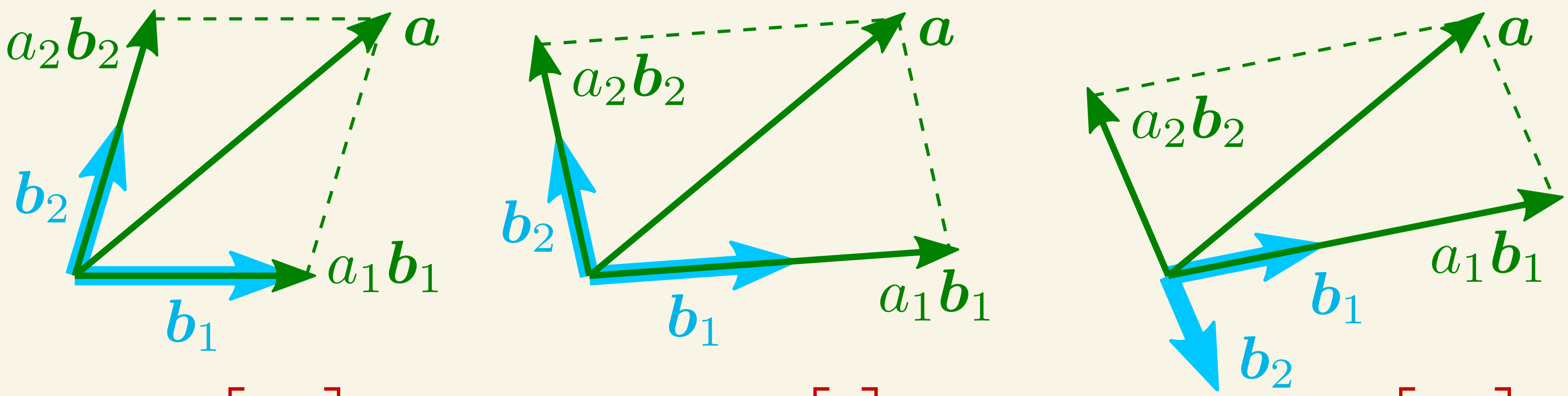
- Observe the difference between the coordinate vector $\underline{\boldsymbol{a}} \in \mathsf{R}^m$, which is a list of real numbers, and the vector it represents, $\boldsymbol{a} \in U$, which could be (almost) anything: force, velocity, ...

$$\underline{\boldsymbol{a}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \in \mathsf{R}^m \qquad \text{but} \qquad \boldsymbol{a} = \sum_{i=1}^{m} a_i \boldsymbol{b}_i \in \mathrm{span}(B) \subseteq U$$

Note: if $m = n$ then $\mathrm{span}(B) = U$.

# Bases and Coordinates (revision)

Some examples (using Euclidean vectors):




$$\underline{\boldsymbol{a}} = \begin{bmatrix} 1.2 \\ 2 \end{bmatrix} \qquad \underline{\boldsymbol{a}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \underline{\boldsymbol{a}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

The Euclidean vector $\boldsymbol{a}$ is the same in all 3 examples, but the coordinate vector $\underline{\boldsymbol{a}}$ varies with the choice of basis vectors.

# Bases and Coordinates (revision)

- If $U$ is Euclidean then it is possible to define a basis with the following property:

$$\boldsymbol{b}_i \cdot \boldsymbol{b}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

  A basis with this property is called an *orthonormal basis*, and it produces a *Cartesian coordinate system*.

- If $B$ is orthonormal then the coordinates of $\boldsymbol{a}$ in $B$ are

$$a_i = \boldsymbol{b}_i \cdot \boldsymbol{a}$$

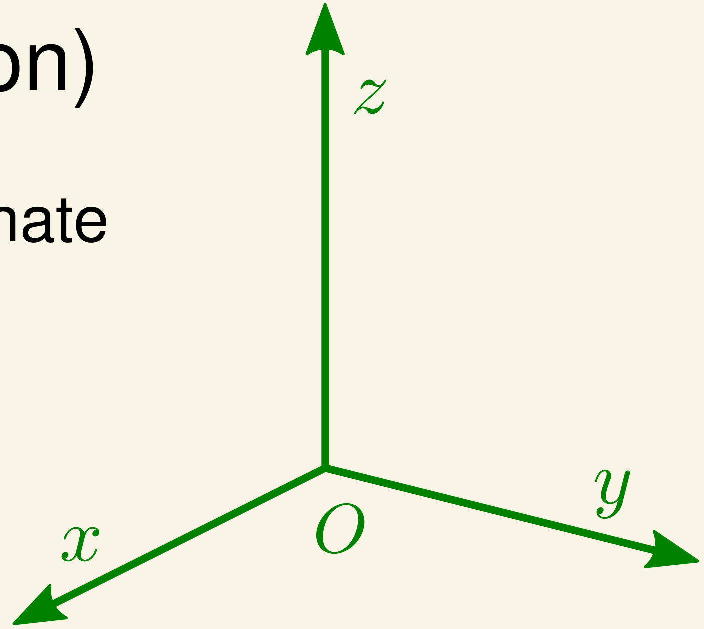
- If coordinate vectors $\underline{\boldsymbol{a}}$ and $\underline{\boldsymbol{c}}$ represent Euclidean vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ in the same Cartesian coordinate system then

$$\boldsymbol{a} \cdot \boldsymbol{c} = \underline{\boldsymbol{a}}^{\mathrm{T}} \underline{\boldsymbol{c}}$$

# Bases and Coordinates (revision)

In 3D space we define a Cartesian coordinate system using a *Cartesian frame*.

This frame defines an origin, $O$, and three directions, $x$, $y$ and $z$.

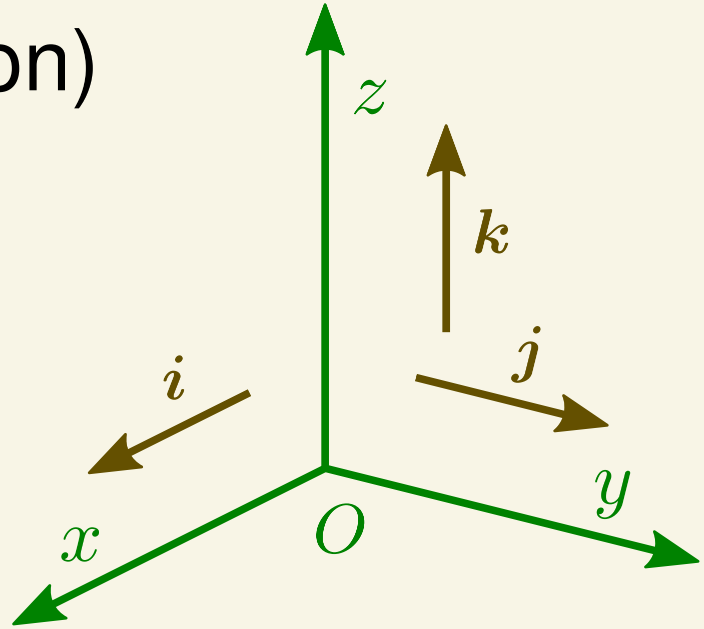



It also defines three directed lines, $Ox$, $Oy$ and $Oz$, which lie on the $x$, $y$ and $z$ axes, respectively. We will be using them later.

# Bases and Coordinates (revision)

Cartesian coordinates can be used to define the

- *position* of a point, or the
- *magnitude and direction* of a Euclidean vector




To accomplish the latter we introduce an orthonormal basis, $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, aligned with the frame's $x$, $y$ and $z$ directions.

$$\underline{\boldsymbol{a}} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \in \mathsf{R}^3 \qquad\qquad \boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \in \mathsf{E}^3$$

Now try question set A
in part1Q.pdf

(answers in part1A.pdf)

# Vector Fields

A vector field is a function that maps each point in a Euclidean (point) space to a Euclidean vector at that point. In effect, it associates a magnitude and a direction with each point in that space. Vector fields can describe a variety of physical phenomena, such as:

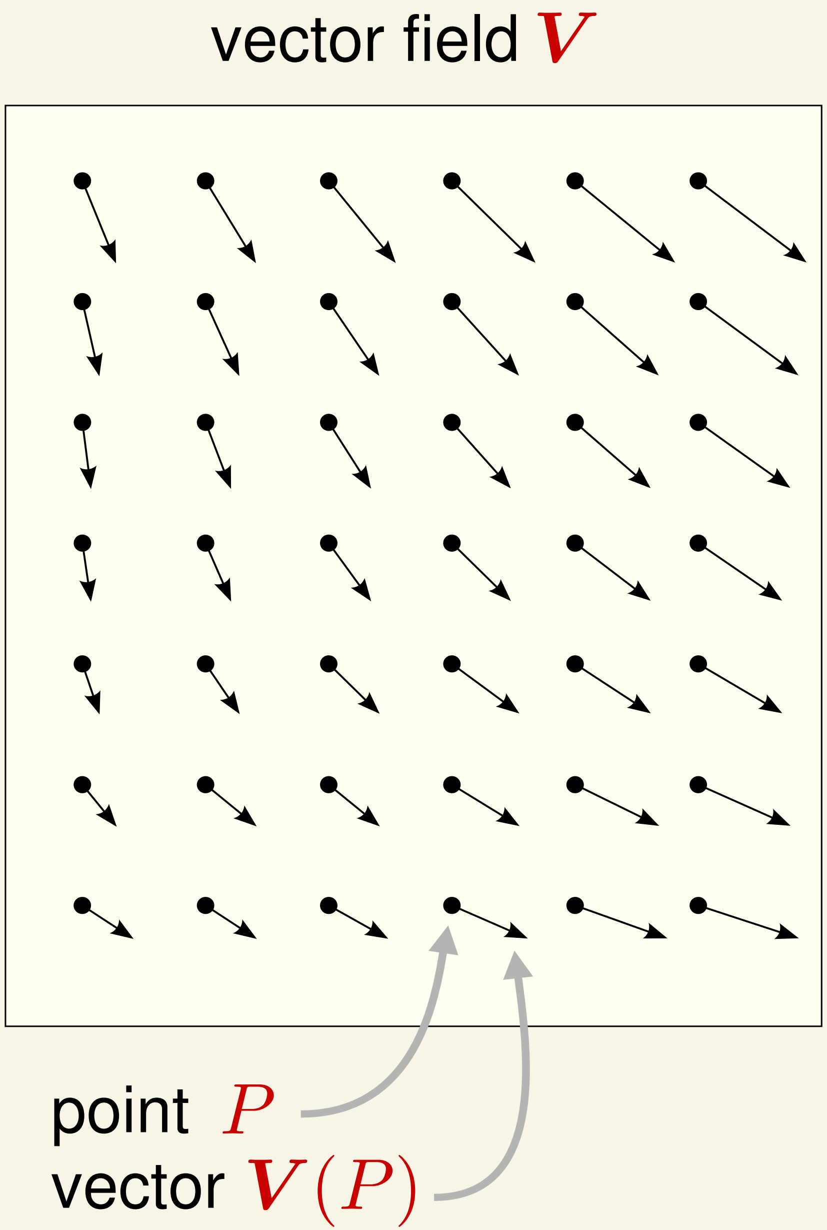
- force fields (gravitational, magnetic, etc.)
- fluid flows
- the velocity of a rigid body

**Note:** A *Euclidean space* is a set of points with a Euclidean distance metric. A *Euclidean **vector** space* is a space of Euclidean vectors. Physical space is a Euclidean space.

# Vector Fields

A vector field can be shown graphically as a collection of arrows emanating from a representative sample of points.

The length and direction of the arrow show the magnitude and direction of the vector.



# Vector Fields

A vector field is itself a vector because it satisfies the formal definition. The two basic operations on vector fields are:

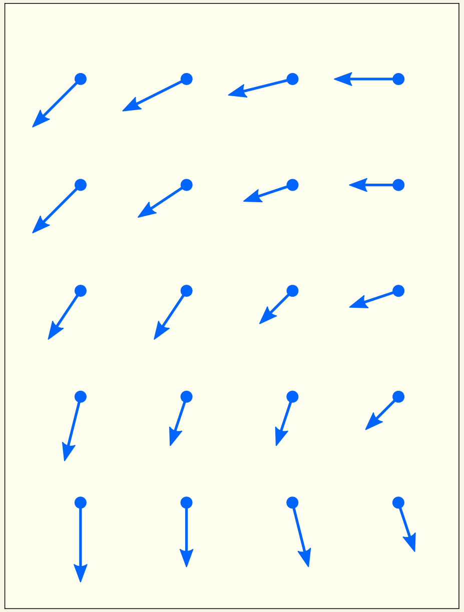
Addition:

$$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2 \quad \text{iff} \quad \boldsymbol{V}(P) = \boldsymbol{V}_1(P) + \boldsymbol{V}_2(P) \quad \text{for all} \quad P$$

Multiplication by a scalar:

$$\boldsymbol{V} = \alpha \boldsymbol{V}_1 \quad \text{iff} \quad \boldsymbol{V}(P) = \alpha \boldsymbol{V}_1(P) \quad \text{for all} \quad P$$

# Vector Fields



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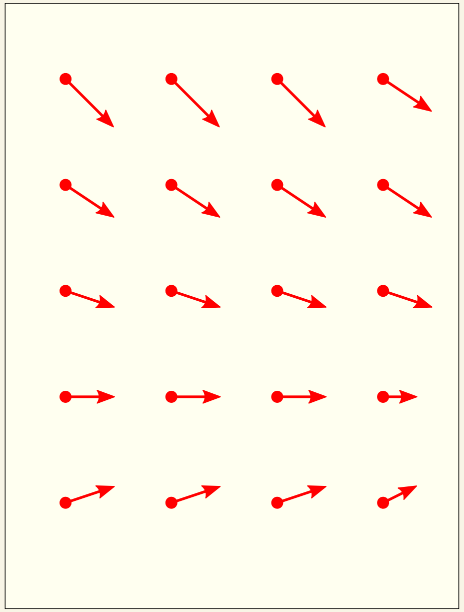

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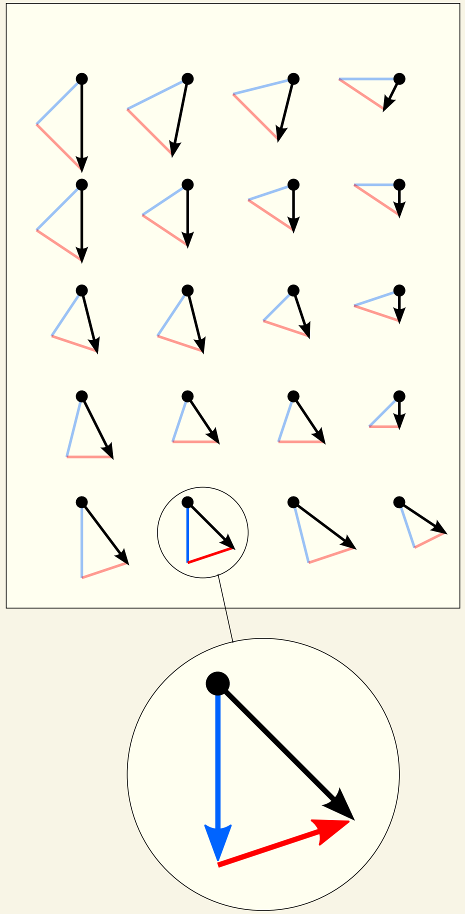

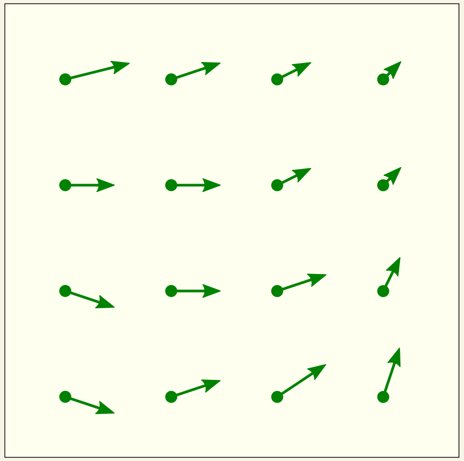

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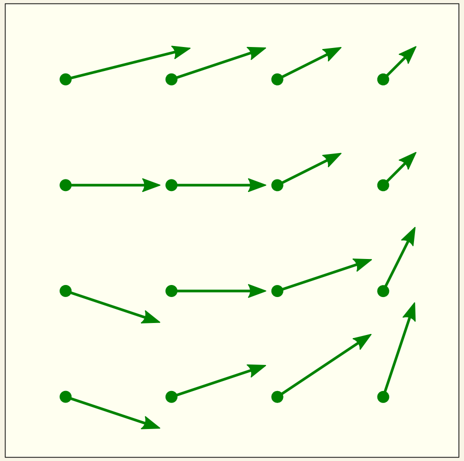
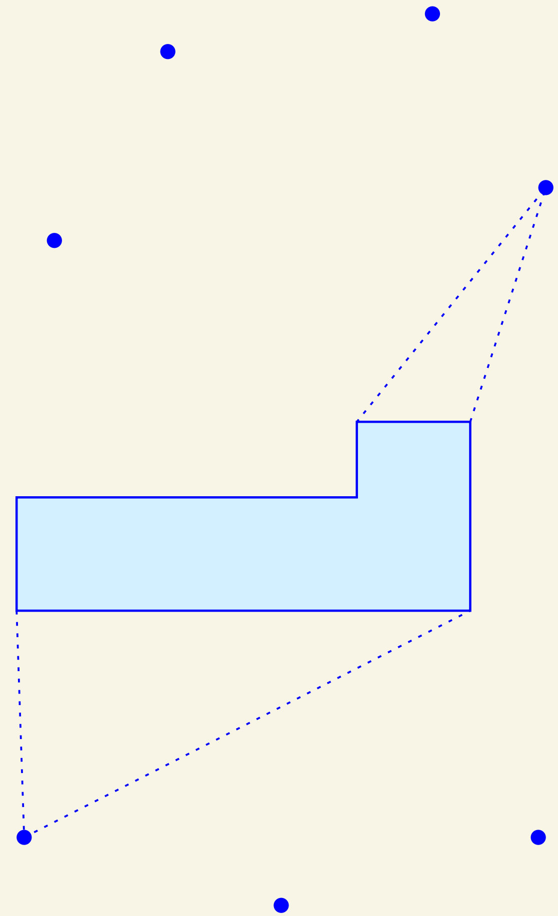


# Body-Fixed Points

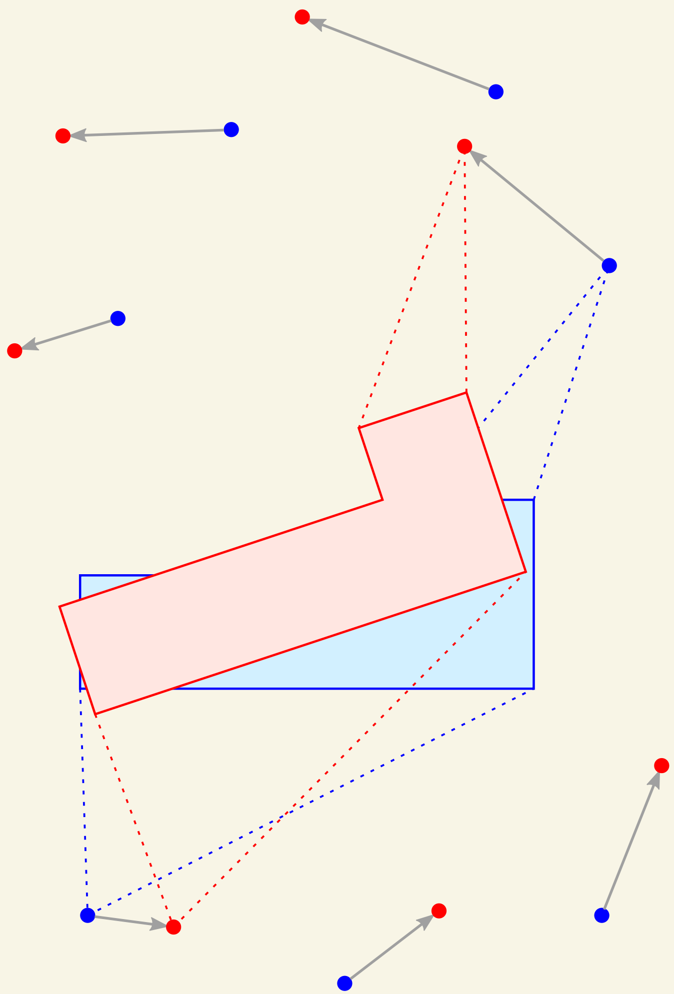
A *body-fixed point* is a point in a fixed location relative to a rigid body.



# Body-Fixed Points

A *body-fixed point* is a point in a fixed location relative to a rigid body.

When the body moves, the point moves with it.

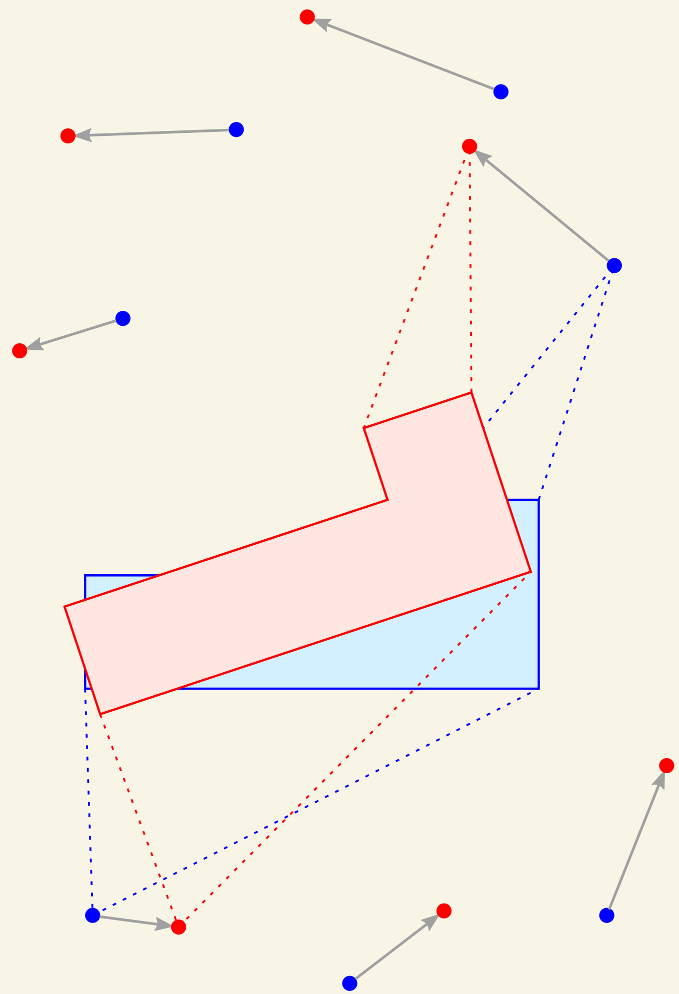


# Body-Fixed Points

A *body-fixed point* is a point in a fixed location relative to a rigid body.

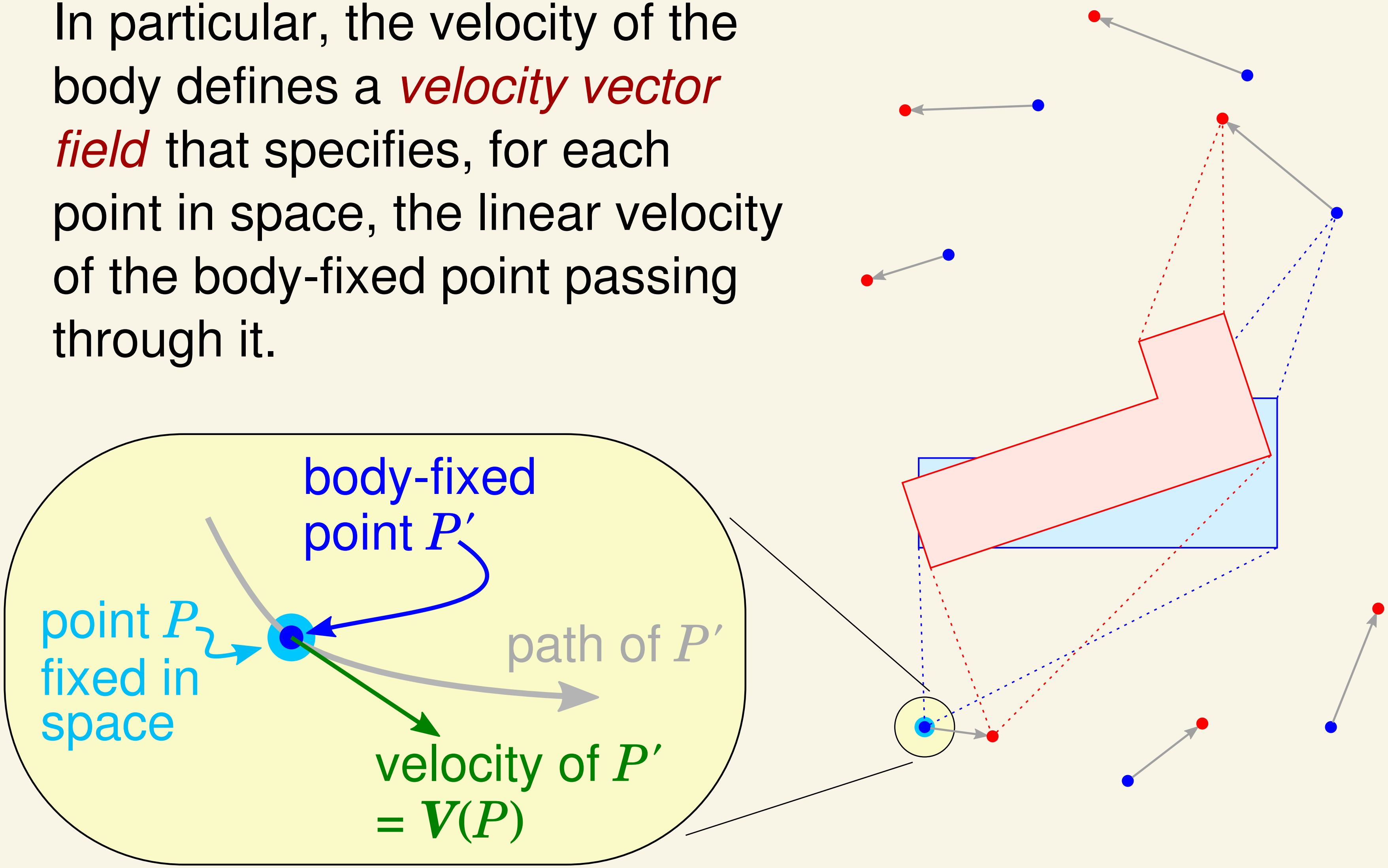
When the body moves, the point moves with it.

If we imagine the whole of space to be filled with body-fixed points, then the motion of the body defines a *vector field*.



# Velocity Vector Field

In particular, the velocity of the body defines a *velocity vector field* that specifies, for each point in space, the linear velocity of the body-fixed point passing through it.

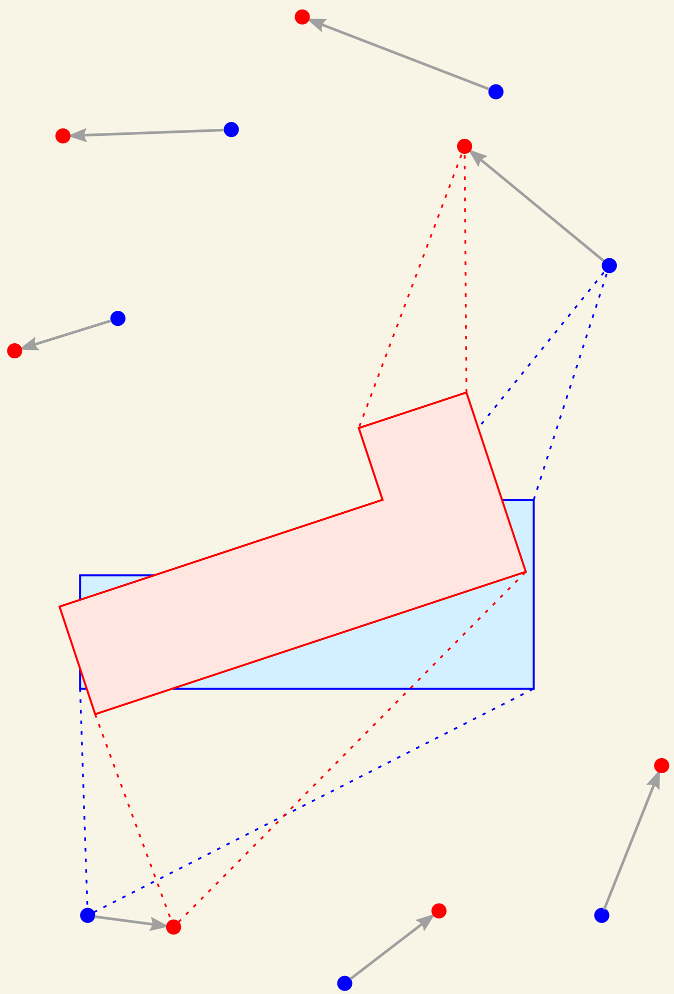


# Velocity Vector Field

In particular, the velocity of the body defines a *velocity vector field* that specifies, for each point in space, the linear velocity of the body-fixed point passing through it.

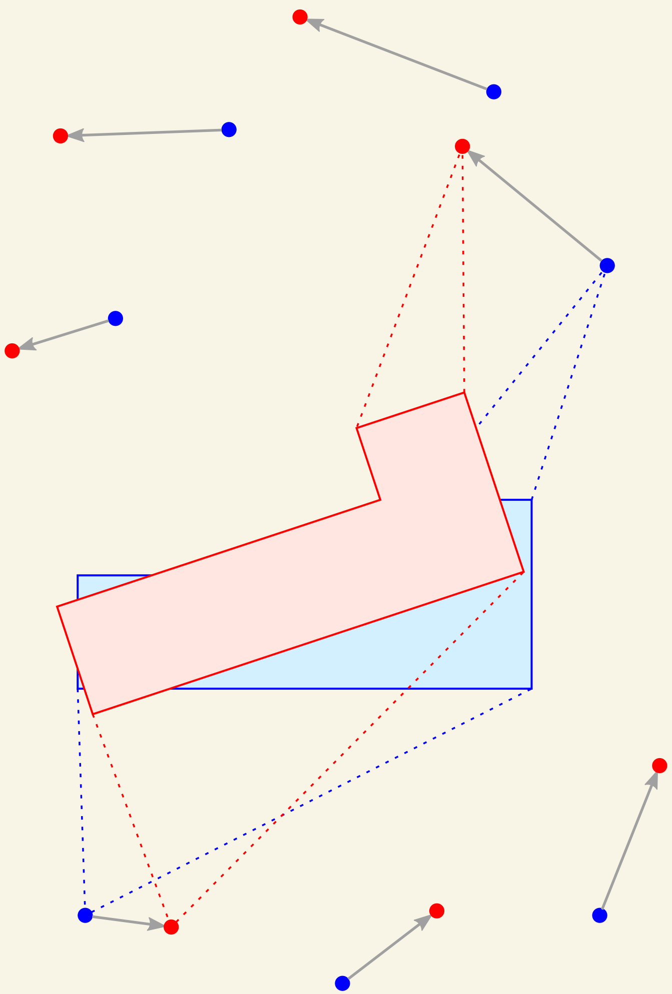
This field provides a complete description of the body's velocity in a way that is independent of whether the body is translating, rotating, or both.

It describes *any* rigid-body velocity.



# Velocity Vector Field = Spatial Velocity

The set of vector fields that describe every possible velocity of a rigid body moving in 3D space forms a *6D vector space*. The elements of this space are *spatial velocity vectors*.



(If the body's motion is restricted to a 2D plane then we get a 3D vector space whose elements are *planar velocity vectors*.)

# Planar Velocity Example

If a rigid body is constrained to move in a plane then there are only two types of velocity it can have:

- pure translation, or
- rotation about a point.

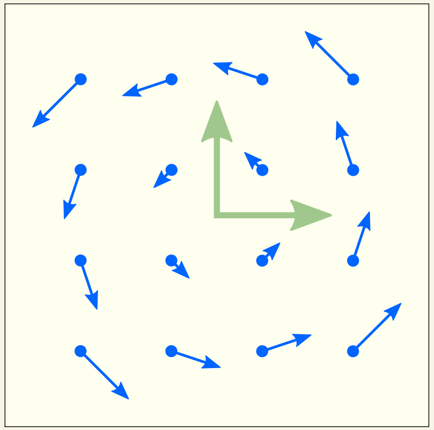
Together they form the 3D space of planar velocity vector fields.

Let us now define a basis on this space . . . .

# Planar Velocity Example

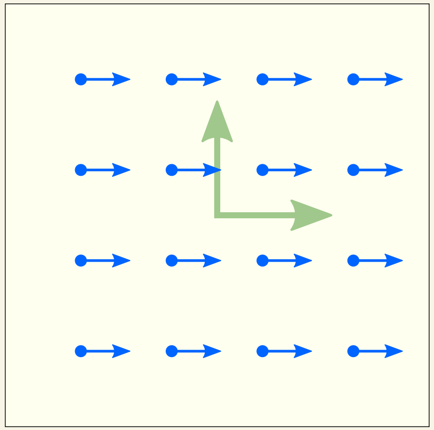
If we place a Cartesian frame anywhere in the plane then we can use it to define the following three basis vectors:

$\boldsymbol{d}_O$



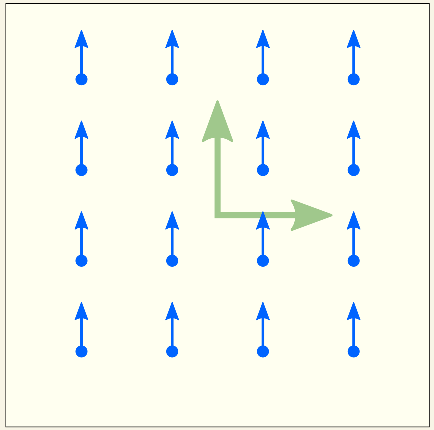
unit rotation
about the origin

$\boldsymbol{d}_x$



unit translation
in the $x$ direction

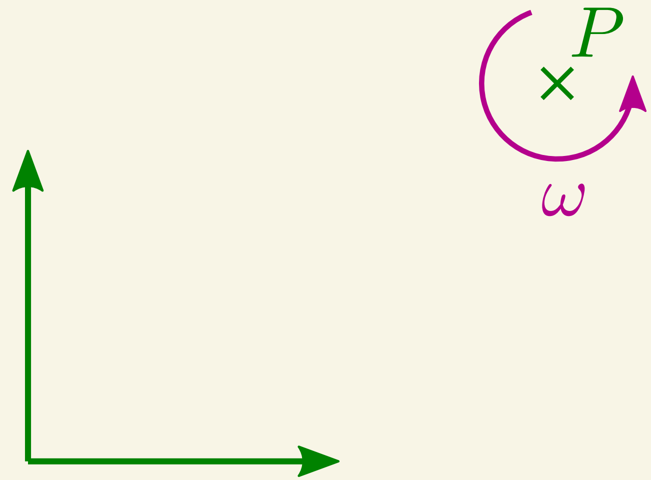
$\boldsymbol{d}_y$



unit translation
in the $y$ direction

# Planar Velocity Example

Suppose that a rigid body is rotating with angular velocity $\omega$ about a point $P$ having coordinates $P_x$ and $P_y$ in our chosen coordinate system.




How do we find the coordinates of this velocity in the basis $\{\boldsymbol{d}_O, \boldsymbol{d}_x, \boldsymbol{d}_y\}$?

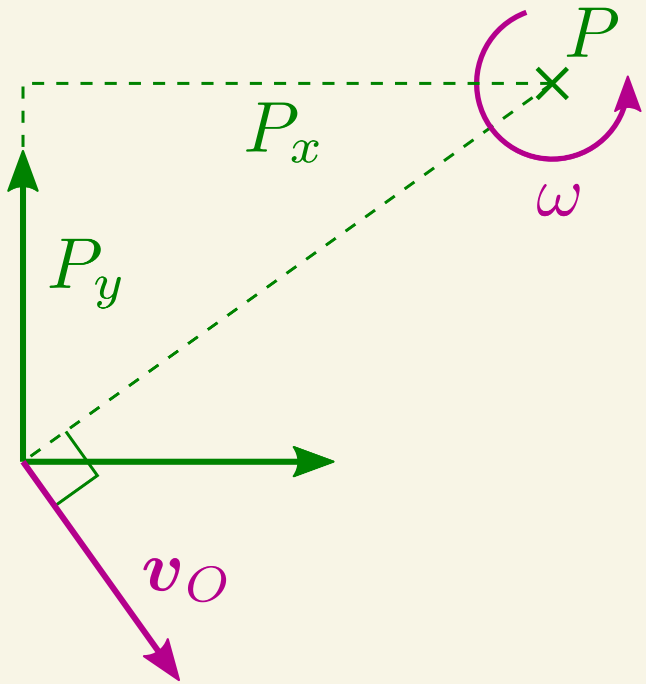
# Planar Velocity Example

Step 1: Calculate the linear velocity of the body-fixed point at the origin:

$$\begin{aligned} \boldsymbol{v}_O &= \omega P_y \boldsymbol{i} - \omega P_x \boldsymbol{j} \\ &= \boldsymbol{V}(O) \end{aligned}$$

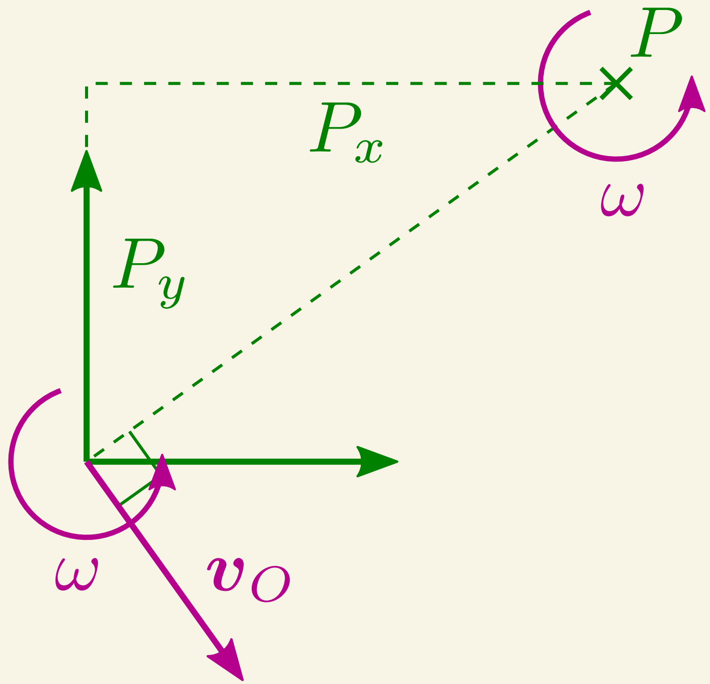
and the corresponding coordinate vector is

$$\underline{\boldsymbol{v}}_O = \begin{bmatrix} \omega P_y \\ -\omega P_x \end{bmatrix}$$



# Planar Velocity Example

Step 2: The velocity of the body can now be expressed as the sum of a rotation of magnitude $\omega$ about the origin and a translation of $\boldsymbol{v}_O$.




These two quantities are $\omega \boldsymbol{d}_O$ and $\omega P_y \boldsymbol{d}_x - \omega P_x \boldsymbol{d}_y$, respectively, so the planar velocity of the rigid body is

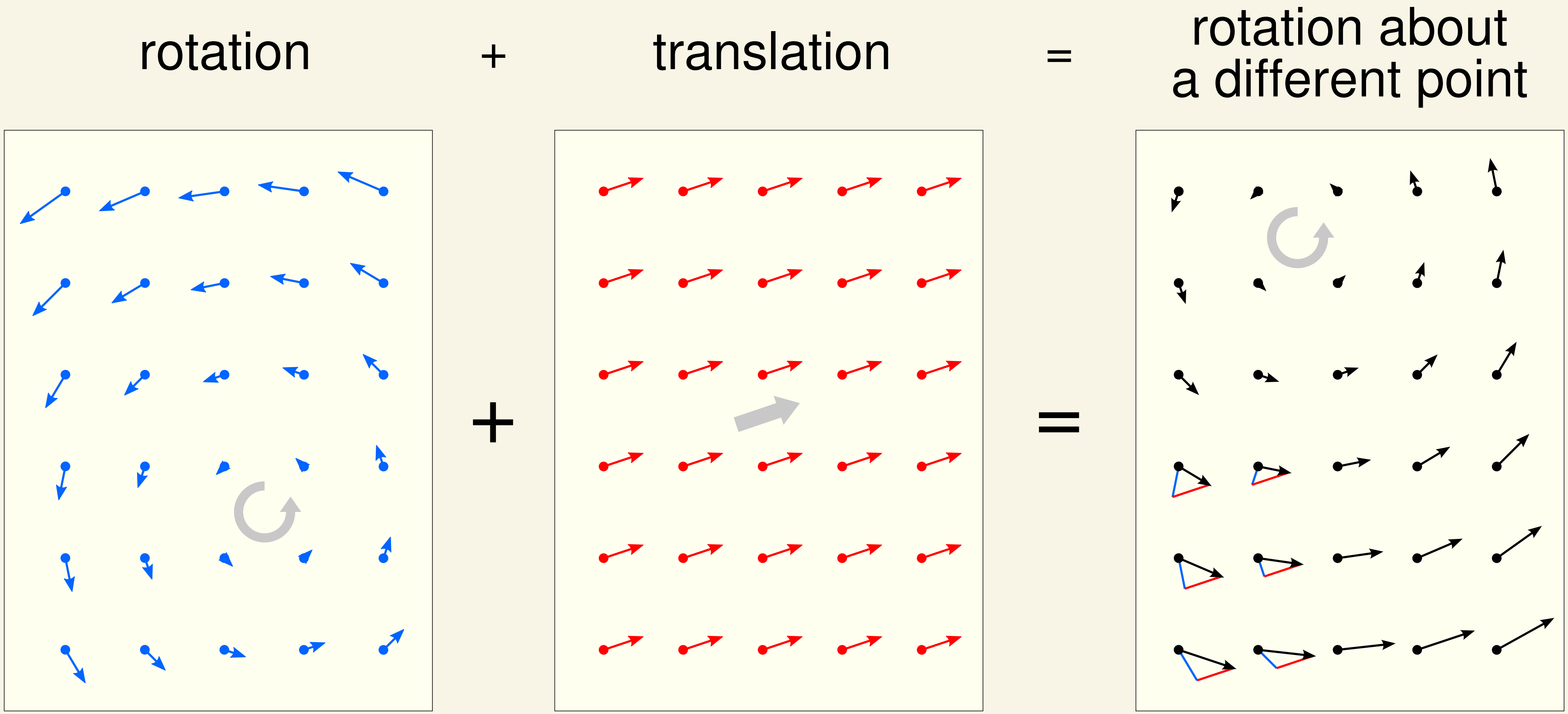
$$\hat{\boldsymbol{v}} = \omega \boldsymbol{d}_O + \omega P_y \boldsymbol{d}_x - \omega P_x \boldsymbol{d}_y$$

(actual velocity)

$$\underline{\hat{\boldsymbol{v}}} = \begin{bmatrix} \omega \\ \omega P_y \\ -\omega P_x \end{bmatrix}$$

(coordinate vector)

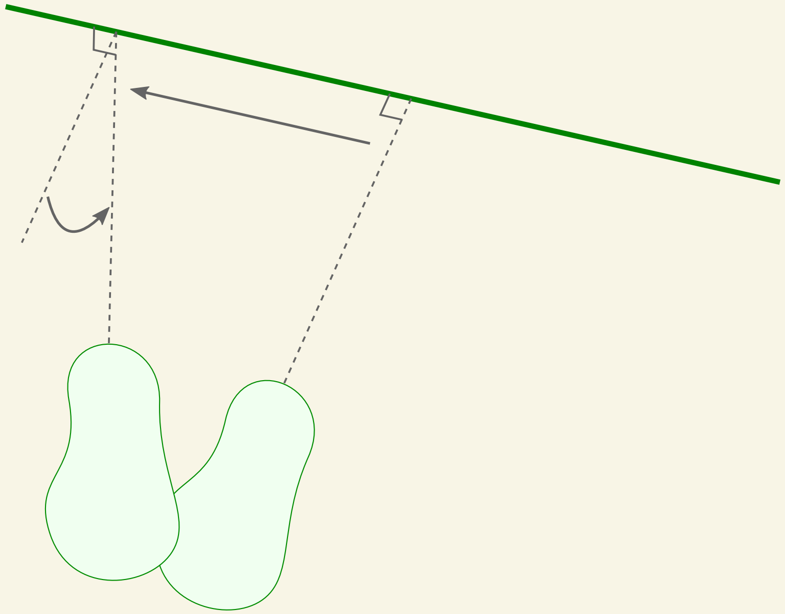
# Addition of Velocity Vector Fields




Adding a translation to a rotation moves the rotation centre in a direction at right angles to the translation.

# Screwing Motion

The most general movement of a rigid body is a *screwing motion* consisting of a translation along and rotation about a particular line in space.



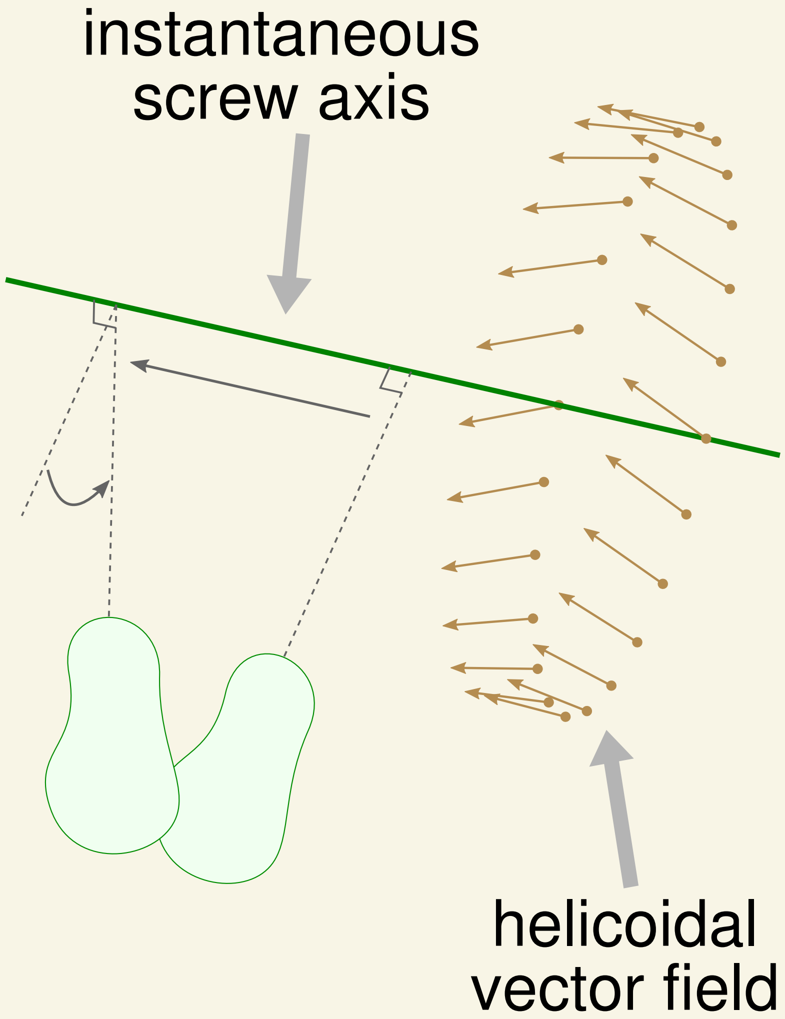
(The study of this kind of motion is the subject of *screw theory*, which is closely related to spatial vector algebra.)

# Screwing Motion

The most general *velocity* of a rigid body is therefore also a screwing motion about a particular line in space.

In this case the line is called the *instantaneous screw axis*, and the velocity vector field is *helicoidal* in shape.

(In screw theory this motion is called a *twist velocity*, or simply a *twist*.)

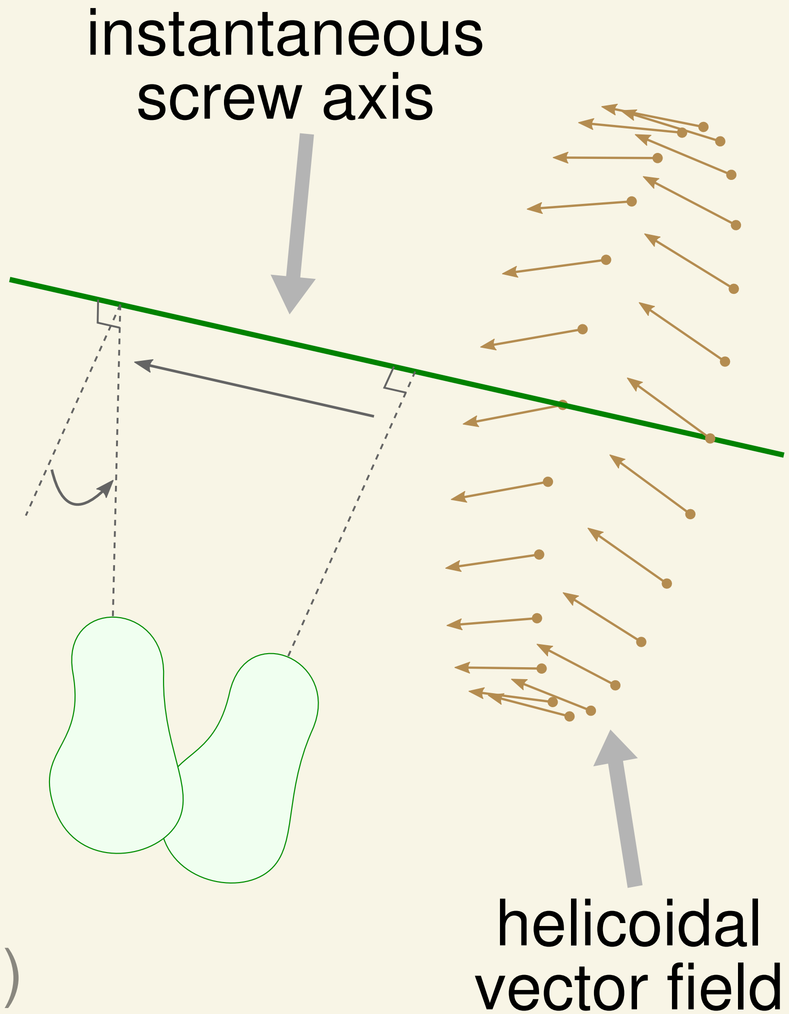


# Screwing Motion

The body's velocity can now be described using two numbers and a line:

- a *linear* velocity magnitude
- an *angular* velocity magnitude
- the instantaneous screw axis

(In screw theory the ratio of the two magnitudes is called the *pitch*.)



# Screwing Motion

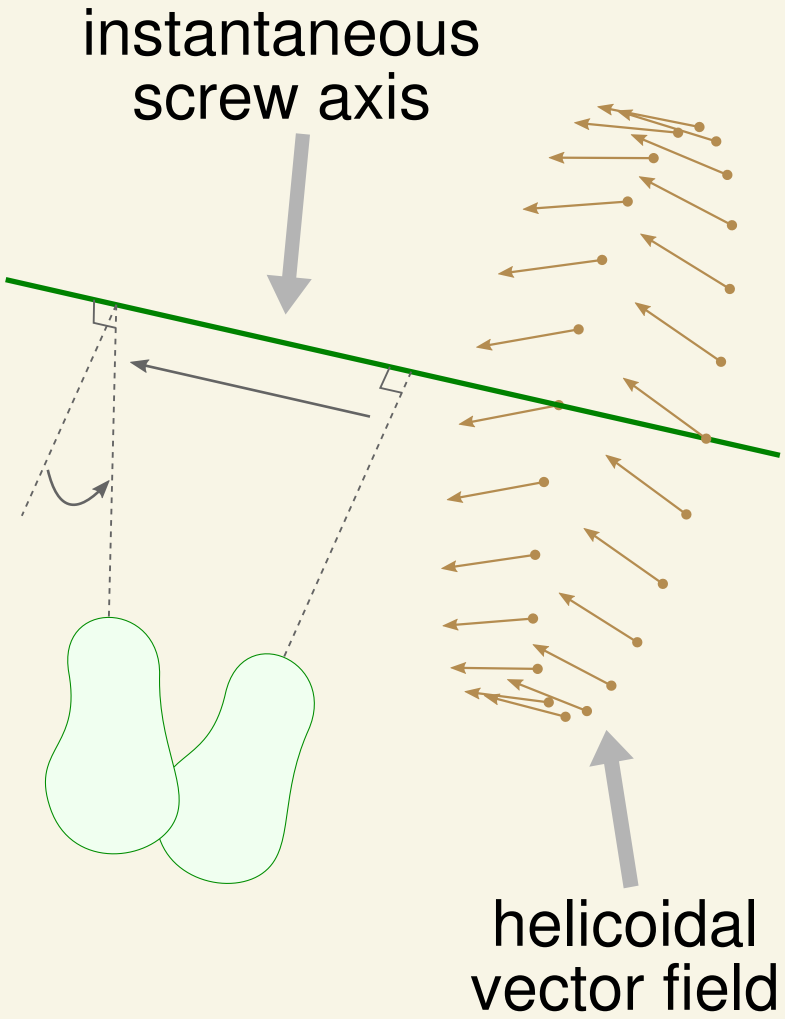
Observe the difference between the velocity of a *particle* and that of a *rigid body*.

particle

- magnitude and direction (3D Euclidean vector)

rigid body

- two magnitudes and a line (6D spatial vector)



# Coordinates

We now have two ways to describe rigid-body velocity:

1. a vector field, or
2. two numbers and a line.

However, all velocities are elements of vector spaces, so we also have a third method:

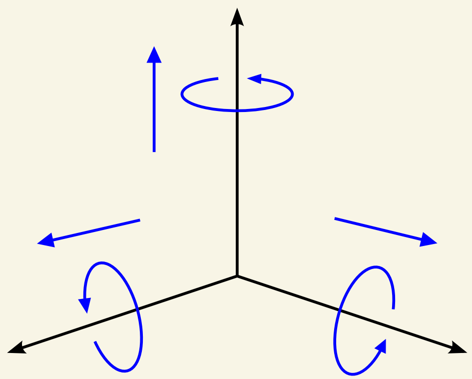
3. a linear combination of *six basis vectors*.

In this case, the velocity is described by a *6D coordinate vector* in the coordinate system defined by the basis vectors.

# Coordinates

The most useful kind of basis consists of the following:

- three unit rotations about the $x$, $y$ and $z$ axes of a Cartesian frame, plus
- three unit translations in the $x$, $y$ and $z$ directions of the same frame.



This is called a *Plücker basis*, and it gives rise to a *Plücker coordinate system*, which we will discuss in more detail later.

# Summary

We have now covered the following items:

- revision of vectors, bases and coordinates;
- vector fields;
- rigid-body velocity treated as a vector field, with velocity in the plane as an example;
- general screwing motion in 3D;
- a preview of Plücker coordinates.

# Now try question set B in part1Q.pdf

(answers in part1A.pdf)