An Introduction to Spatial (6D) Vectors

Answers Part 3

Topics: Plücker Coordinates, Differentiation and Acceleration
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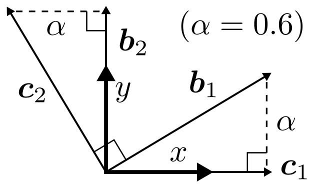
Answer A1

(a) $\boldsymbol{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{c}_2 = \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}$

Although this problem is easily solved in an ad-hoc manner, there is also a systematic method that generalizes to n dimensions. Let $\boldsymbol{B}$ be the 2×2 matrix $[\boldsymbol{b}_1\ \boldsymbol{b}_2]$, and let $\boldsymbol{C} = [\boldsymbol{c}_1\ \boldsymbol{c}_2]$. The reciprocity condition is then $\boldsymbol{B}\,\boldsymbol{C}^{\mathrm{T}} = \mathbf{1}$, which implies $\boldsymbol{C}^{\mathrm{T}} = \boldsymbol{B}^{-1}$. So $\boldsymbol{c}_i$ is simply row i of $\boldsymbol{B}^{-1}$.

(b) 

Answer A2

(a) $\boldsymbol{d}_x$, $\boldsymbol{d}_y$ and $\boldsymbol{d}_z$ are the same in both bases because these vectors depend only on the x, y and z directions, which are the same for both coordinate frames. We also have $\boldsymbol{d}_{Qy} = \boldsymbol{d}_{Oy}$ because $Qy = Oy$. Thus, the only two vectors that are different in D_Q are

$$\boldsymbol{d}_{Qx} = \boldsymbol{d}_{Ox} - l\,\boldsymbol{d}_z \qquad \text{and} \qquad \boldsymbol{d}_{Qz} = \boldsymbol{d}_{Oz} + l\,\boldsymbol{d}_x\,.$$

Tip: A quick way to work out the answer is to imagine a rigid body performing the rotation you want to represent, and ask what happens to the body-fixed point at O. For example, if the body performs a rotation about Qx at unit angular velocity then the body-fixed point at O will move straight down with a linear velocity magnitude of l, so $\boldsymbol{d}_{Qx} = \boldsymbol{d}_{Ox} - l\,\boldsymbol{d}_z$.

(b) The coordinates ω_x, ω_y and ω_z are the same in both vectors. To obtain expressions for the linear coordinates, we use the formula $\boldsymbol{v}_Q = \boldsymbol{v}_O - \overrightarrow{OQ} \times \boldsymbol{\omega}$ with $\overrightarrow{OQ} = [0\ l\ 0]^{\mathrm{T}}$. This gives

$$\begin{aligned} v_{Qx} &= v_{Ox} - l\,\omega_z \\ v_{Qy} &= v_{Oy} \\ v_{Qz} &= v_{Oz} + l\,\omega_x \end{aligned}$$

(c) $$\begin{aligned} &\omega_x \boldsymbol{d}_{Qx} + \omega_y \boldsymbol{d}_{Qy} + \omega_z \boldsymbol{d}_{Qz} + v_{Qx}\boldsymbol{d}_x + v_{Qy}\boldsymbol{d}_y + v_{Qz}\boldsymbol{d}_z \\ &= \omega_x(\boldsymbol{d}_{Ox} - l\boldsymbol{d}_z) + \omega_y \boldsymbol{d}_{Oy} + \omega_z(\boldsymbol{d}_{Oz} + l\boldsymbol{d}_x) + (v_{Ox} - l\omega_z)\boldsymbol{d}_x + v_{Oy}\boldsymbol{d}_y + (v_{Oz} + l\omega_x)\boldsymbol{d}_z \\ &= \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + v_{Ox}\boldsymbol{d}_x + v_{Oy}\boldsymbol{d}_y + v_{Oz}\boldsymbol{d}_z\,. \end{aligned}$$

Answer B1

$$\boldsymbol{a}_1 = \boldsymbol{s}_1\,\ddot{q}_1 + \dot{\boldsymbol{s}}_1\,\dot{q}_1 = \boldsymbol{s}_1\,\ddot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{a}_2 &= \boldsymbol{a}_1 + \boldsymbol{s}_2\,\ddot{q}_2 + \dot{\boldsymbol{s}}_2\,\dot{q}_2 \\ &= \boldsymbol{a}_1 + \boldsymbol{s}_2\,\ddot{q}_2 + \boldsymbol{v}_1 \times \boldsymbol{s}_2\,\dot{q}_2 \\ &= \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_2 \\ 0 \\ -\ddot{q}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \\ 0 \\ -\dot{q}_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 + \ddot{q}_2 \\ 0 \\ -\ddot{q}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dot{q}_1\,\dot{q}_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_1 + \ddot{q}_2 \\ \dot{q}_1\,\dot{q}_2 \\ -\ddot{q}_2 \\ 0 \end{bmatrix} \end{aligned}$$

Answer B2

Let C denote the position of a point on the central axis of the cylinder. The coordinates of C are then $(0, y_0 + v\,t, r)$, where y_0 is the y coordinate of C at $t = 0$. The angular velocity of the cylinder is $\boldsymbol{\omega} = [\,-v/r\ 0\ 0\,]^{\mathrm{T}}$, and the linear velocity at C is $\boldsymbol{v}_C = [\,0\ v\ 0\,]^{\mathrm{T}}$. The linear velocity at O is therefore

$$\boldsymbol{v}_O = \boldsymbol{v}_C + \overrightarrow{OC} \times \boldsymbol{\omega} = \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y_0 + v\,t \\ r \end{bmatrix} \times \begin{bmatrix} -v/r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ (y_0 + v\,t)\,v/r \end{bmatrix}.$$

Let $\hat{\boldsymbol{a}}_O$ be the coordinate vector expressing the spatial acceleration of the cylinder at O. As O is a fixed point in space, $\hat{\boldsymbol{a}}_O$ is just the componentwise derivative of the spatial velocity, $\hat{\boldsymbol{v}}_O$:

$$\hat{\boldsymbol{a}}_O = \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\boldsymbol{v}}_O = \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} -v/r \\ 0 \\ 0 \\ 0 \\ 0 \\ (y_0 + v\,t)\,v/r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v^2/r \end{bmatrix}.$$

Note: if we wish to perform this calculation at the moving point C, instead of the fixed point O, then we must calculate $\hat{\boldsymbol{a}}_C$ using the formula for differentiation in a moving Plücker coordinate system.