An Introduction to Spatial (6D) Vectors

Answers Part 4

Topics: Dynamics
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Answer A1

$$
\begin{bmatrix} \bar{\boldsymbol{I}}_O & m\vec{\boldsymbol{c}}\times \\ m\vec{\boldsymbol{c}}\times^{\mathrm{T}} & m\boldsymbol{1} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\omega}} \\ \vec{\boldsymbol{v}}_O \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{I}}_C\vec{\boldsymbol{\omega}} - m\vec{\boldsymbol{c}}\times\vec{\boldsymbol{c}}\times\vec{\boldsymbol{\omega}} + m\vec{\boldsymbol{c}}\times\vec{\boldsymbol{v}}_O \\ -m\vec{\boldsymbol{c}}\times\vec{\boldsymbol{\omega}} + m\vec{\boldsymbol{v}}_O \end{bmatrix}
$$

$$
= \begin{bmatrix} \vec{\boldsymbol{h}}_C + \vec{\boldsymbol{c}}\times(m\vec{\boldsymbol{v}}_O - m\vec{\boldsymbol{c}}\times\vec{\boldsymbol{\omega}}) \\ m(\vec{\boldsymbol{v}}_O - \vec{\boldsymbol{c}}\times\vec{\boldsymbol{\omega}}) \end{bmatrix}
$$

$$
= \begin{bmatrix} \vec{\boldsymbol{h}}_C + \vec{\boldsymbol{c}}\times m\vec{\boldsymbol{v}}_C \\ m\vec{\boldsymbol{v}}_C \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{h}}_C + \vec{\boldsymbol{c}}\times\vec{\boldsymbol{h}} \\ \vec{\boldsymbol{h}} \end{bmatrix} = \begin{bmatrix} \vec{\boldsymbol{h}}_O \\ \vec{\boldsymbol{h}} \end{bmatrix}
$$

Answer A2

Starting with the formula $\boldsymbol{I}_B = {}^B\boldsymbol{X}_A^* \boldsymbol{I}_A {}^A\boldsymbol{X}_B$ on slide 9, simply invert both sides:

$$
\begin{aligned}
\boldsymbol{I}_B^{-1} &= ({}^B\boldsymbol{X}_A^* \boldsymbol{I}_A {}^A\boldsymbol{X}_B)^{-1} \\
&= ({}^A\boldsymbol{X}_B)^{-1} \boldsymbol{I}_A^{-1} ({}^B\boldsymbol{X}_A^*)^{-1} \\
&= {}^B\boldsymbol{X}_A \boldsymbol{I}_A^{-1} {}^A\boldsymbol{X}_B^*
\end{aligned}
$$

Answer A3

$\boldsymbol{a}_B = \boldsymbol{I}_B^{-1}\boldsymbol{f}$ and $\boldsymbol{a}_A = -\boldsymbol{I}_A^{-1}\boldsymbol{f}$; so $\boldsymbol{a}_{\mathrm{rel}} = \boldsymbol{a}_B - \boldsymbol{a}_A = (\boldsymbol{I}_A^{-1} + \boldsymbol{I}_B^{-1})\boldsymbol{f}$. Therefore

$$
\boldsymbol{I}_{\mathrm{rel}}^{-1} = \boldsymbol{I}_A^{-1} + \boldsymbol{I}_B^{-1}\,.
$$

The relative inertia is the harmonic sum (the inverse of the sum of the inverses) of the inertias of the participating bodies, whereas the inertia of a composite body is the ordinary sum of the inertias of its parts.

Answer A4

$$
\begin{bmatrix} \vec{\boldsymbol{n}}_C \\ \vec{\boldsymbol{f}} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{I}}_C & \boldsymbol{0} \\ \boldsymbol{0} & m\,\boldsymbol{1} \end{bmatrix} \begin{bmatrix} \dot{\vec{\boldsymbol{\omega}}} \\ \ddot{\vec{\boldsymbol{c}}} - \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{v}}_C \end{bmatrix} + \begin{bmatrix} \vec{\boldsymbol{\omega}}\times & \vec{\boldsymbol{v}}_C\times \\ \boldsymbol{0} & \vec{\boldsymbol{\omega}}\times \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{I}}_C & \boldsymbol{0} \\ \boldsymbol{0} & m\,\boldsymbol{1} \end{bmatrix} \begin{bmatrix} \vec{\boldsymbol{\omega}} \\ \vec{\boldsymbol{v}}_C \end{bmatrix}
$$

$$
= \begin{bmatrix} \bar{\boldsymbol{I}}_C\,\dot{\vec{\boldsymbol{\omega}}} \\ m\,\ddot{\vec{\boldsymbol{c}}} - m\,\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{v}}_C \end{bmatrix} + \begin{bmatrix} \vec{\boldsymbol{\omega}}\times\bar{\boldsymbol{I}}_C\,\vec{\boldsymbol{\omega}} + m\,\vec{\boldsymbol{v}}_C\times\vec{\boldsymbol{v}}_C \\ m\,\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{v}}_C \end{bmatrix}
$$

$$
= \begin{bmatrix} \bar{\boldsymbol{I}}_C\,\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{\omega}}\times\bar{\boldsymbol{I}}_C\,\vec{\boldsymbol{\omega}} \\ m\,\ddot{\vec{\boldsymbol{c}}} \end{bmatrix}.
$$

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Answer B1

Substitute $\boldsymbol{a} = \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}$ into the equation of motion:

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}(\boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}) + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}\,.$$

Find $\dot{\boldsymbol{\alpha}}$:

$$\begin{aligned}\boldsymbol{I}\boldsymbol{S}\dot{\boldsymbol{\alpha}} &= \boldsymbol{f} + \boldsymbol{f}_c - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}\\ \boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\dot{\boldsymbol{\alpha}} &= \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v})\\ \dot{\boldsymbol{\alpha}} &= (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v})\,.\end{aligned}$$

Substitute this expression for $\dot{\boldsymbol{\alpha}}$ back into $\boldsymbol{a} = \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}$:

$$\boldsymbol{a} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}) + \dot{\boldsymbol{S}}\boldsymbol{\alpha}\,.$$

This equation can be expressed in the form

$$\boldsymbol{a} = \boldsymbol{\Phi}\boldsymbol{f} + \boldsymbol{b}$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{b}$ are the apparent inverse inertia and bias acceleration of the constrained body, respectively, and are given by

$$\boldsymbol{\Phi} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$$

and

$$\boldsymbol{b} = \dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{\Phi}(\boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v})\,.$$

Answer B2

$$\begin{aligned}\boldsymbol{a} \;=\; \dot{\boldsymbol{v}} \;&=\; \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{S}\boldsymbol{\alpha})\\ &=\; \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}\\ &=\; \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \boldsymbol{v}\times\boldsymbol{S}\boldsymbol{\alpha}\\ &=\; \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \boldsymbol{v}\times\boldsymbol{v} \;=\; \boldsymbol{S}\dot{\boldsymbol{\alpha}}\end{aligned}$$

$\boldsymbol{T}^{\mathrm{T}}\boldsymbol{S} = \boldsymbol{0}$ implies $\dot{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{S} + \boldsymbol{T}^{\mathrm{T}}\dot{\boldsymbol{S}} = \boldsymbol{0}$, which in turn implies $\dot{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{\alpha} + \boldsymbol{T}^{\mathrm{T}}\dot{\boldsymbol{S}}\boldsymbol{\alpha} = \boldsymbol{0}$. But we already know that $\dot{\boldsymbol{S}}\boldsymbol{\alpha} = \boldsymbol{0}$, so

$$\dot{\boldsymbol{T}}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{0}$$

Answer B3

The equation for $\dot{\boldsymbol{\alpha}}$ obtained in the answer to B1 is

$$\dot{\boldsymbol{\alpha}} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v})\,.$$

Substituting $\ddot{\boldsymbol{q}}$ and $\boldsymbol{\tau}$ gives

$$\ddot{\boldsymbol{q}} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}(\boldsymbol{\tau} - \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}))\,.$$

Rearranging this equation into the form $\boldsymbol{\tau} = \boldsymbol{H}\ddot{\boldsymbol{q}} + \boldsymbol{C}$ gives

$$\begin{aligned}\boldsymbol{H} &= \boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\\ \boldsymbol{C} &= \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v})\end{aligned}$$

Answer B4

$T = \frac{1}{2}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{v}$ by definition (slide 8). Substituting $\boldsymbol{v} = \boldsymbol{S}\dot{\boldsymbol{q}}$ gives

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\dot{\boldsymbol{q}} = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{H}\dot{\boldsymbol{q}}$$

(This is in fact a defining property of $\boldsymbol{H}$.)

Answer C1

Kinetic Energy:

$$\begin{aligned}
&\boldsymbol{v}_0 = \boldsymbol{0} \\
&\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i\dot{q}_i \\
&T_i = \tfrac{1}{2}\boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{I}_i\boldsymbol{v}_i + \textstyle\sum_{j\in\mu(i)} T_j \\
&T_{\mathrm{tot}} = T_1
\end{aligned}$$

Or you could replace the last two lines with $T_{\mathrm{tot}} = \frac{1}{2}\sum_i \boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{I}_i\boldsymbol{v}_i$.

Momentum:

$$\begin{aligned}
&\boldsymbol{v}_0 = \boldsymbol{0} \\
&\boldsymbol{v}_i = \boldsymbol{v}_{\lambda(i)} + \boldsymbol{s}_i\dot{q}_i \\
&\boldsymbol{h}_i = \boldsymbol{I}_i\boldsymbol{v}_i + \textstyle\sum_{j\in\mu(i)} \boldsymbol{h}_j \\
&\boldsymbol{h}_{\mathrm{tot}} = \boldsymbol{h}_1
\end{aligned}$$

where $\boldsymbol{h}_i$ is the total momentum of the subtree consisting of link i and all of its descendants.