An Introduction to Spatial (6D) Vectors and Their Use in Robot Dynamics

Part 2: Motion and Force

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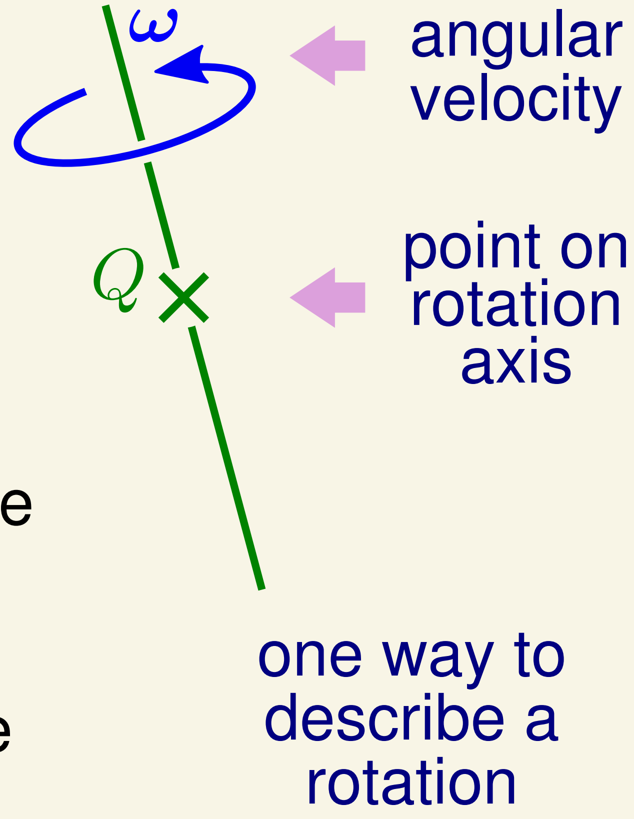
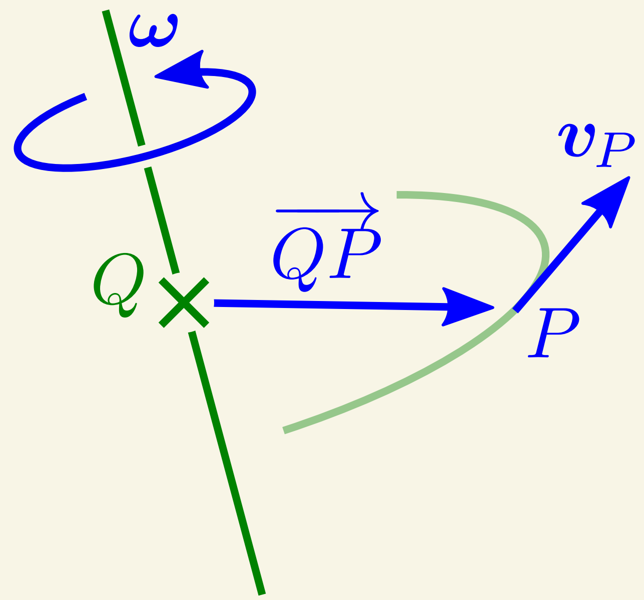
Rotation

- A rotation in 3D is a turning motion about a line in space, which is the axis of rotation.
- So a rotational velocity has a magnitude and a line.
- An angular velocity vector can describe the rate of turning and the direction of the line, but not its position in space.
- The latter can be specified by identifying any one point on the line.



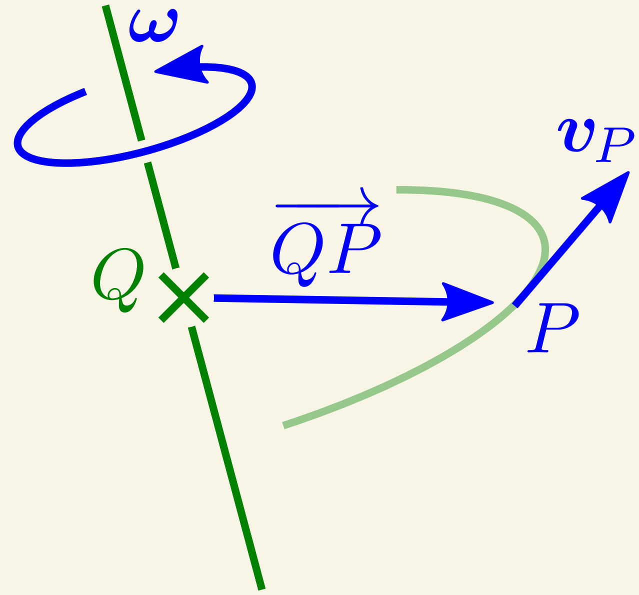
Rotation

Let $\boldsymbol{V}_Q$ be the velocity vector field defined by $\boldsymbol{\omega}$ and Q. The linear velocity of the body-fixed point at any position P is then



$$\boldsymbol{V}_Q(P) = \boldsymbol{\omega} \times \overrightarrow{QP} = \boldsymbol{v}_P$$

Rotation



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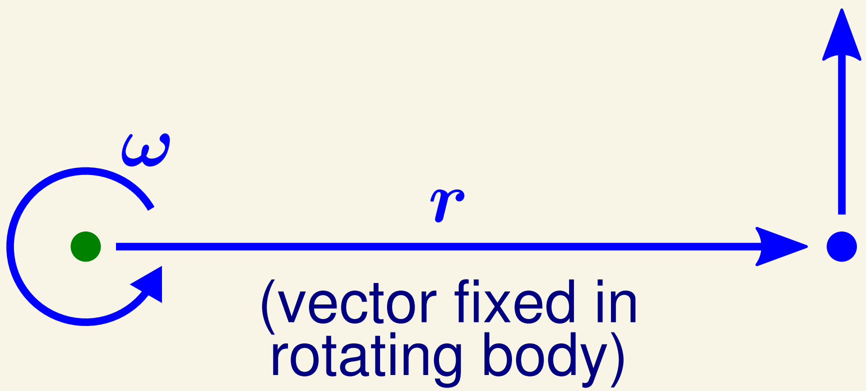
$$\boldsymbol{V}_Q(P) = \boldsymbol{\omega} \times \overrightarrow{QP} = \boldsymbol{v}_P$$

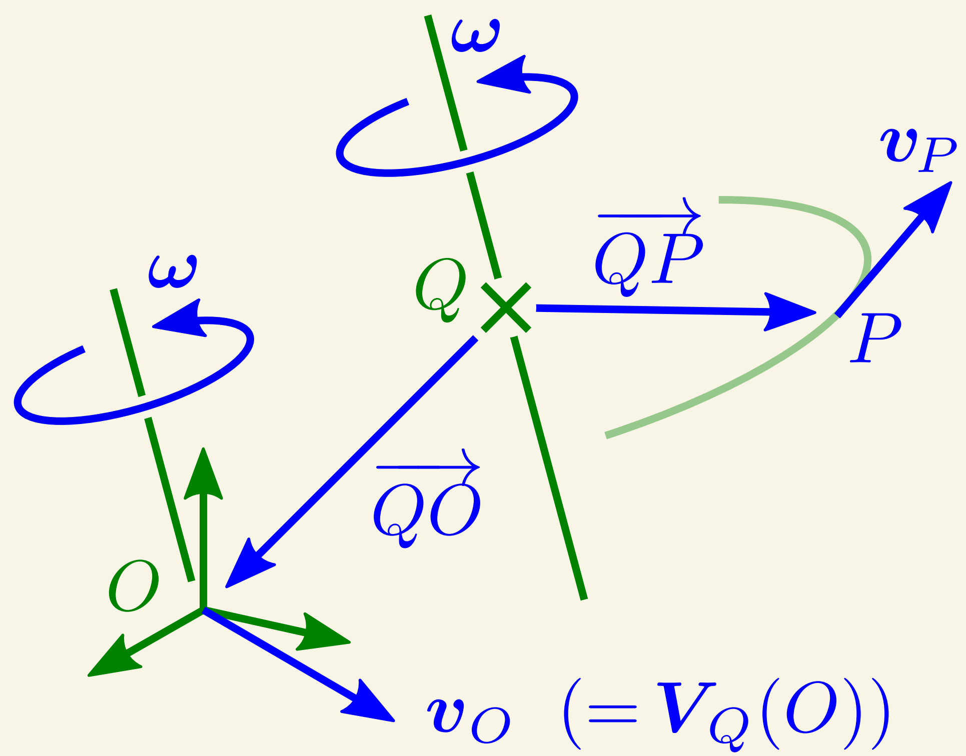
Observe the role of the cross product in calculating linear velocity from rotational velocity.



$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$$

Rotation

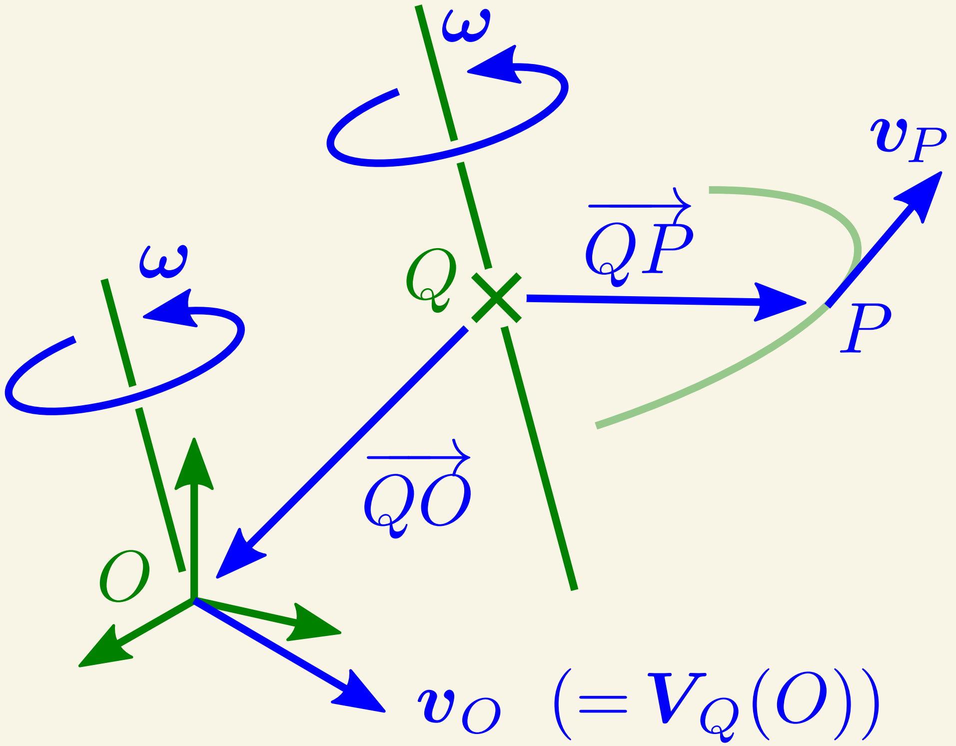
Now introduce a coordinate frame with origin at O and a second velocity field, $\boldsymbol{V}_O$, having the same magnitude and direction as $\boldsymbol{V}_Q$ but with its rotation axis passing through O.



Rotation

We now have

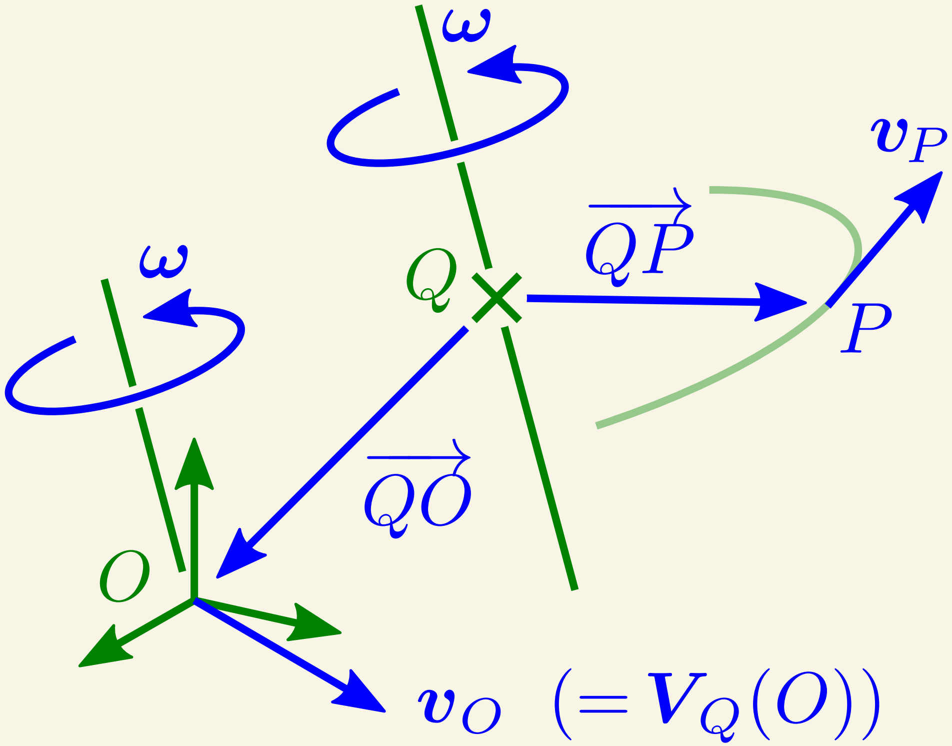
$$\begin{aligned}
\boldsymbol{V}_O(P) &= \boldsymbol{\omega} \times \overrightarrow{OP} \\
&= \boldsymbol{\omega} \times (\overrightarrow{OQ} + \overrightarrow{QP}) \\
&= \boldsymbol{V}_Q(P) + \boldsymbol{\omega} \times \overrightarrow{OQ} \\
&= \boldsymbol{V}_Q(P) - \boldsymbol{\omega} \times \overrightarrow{QO} \\
&= \boldsymbol{V}_Q(P) - \boldsymbol{V}_Q(O)
\end{aligned}$$



So

$$\boldsymbol{V}_Q(P) = \boldsymbol{V}_O(P) + \boldsymbol{V}_Q(O)$$

Rotation

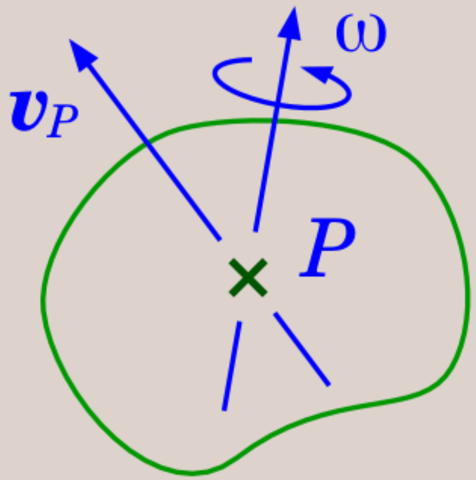


A general rule:

$$\boldsymbol{V}_Q(P) = \boldsymbol{V}_O(P) + \boldsymbol{v}_O$$

A rotational velocity about a line anywhere in space is equivalent to the sum of a parallel rotation about a line passing through the origin and a translational velocity equal to the velocity of the body-fixed point at the origin.

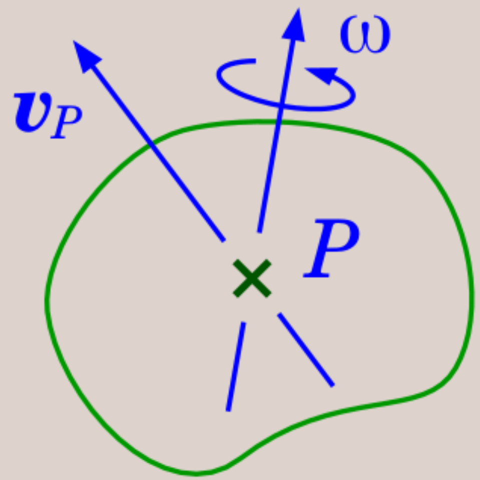
Velocity



The velocity of a rigid body can be described by

1. choosing a point, P, in the body
2. specifying the linear velocity, $\boldsymbol{v}_P$, of that point, and
3. specifying the angular velocity, ω, of the body as a whole

Velocity



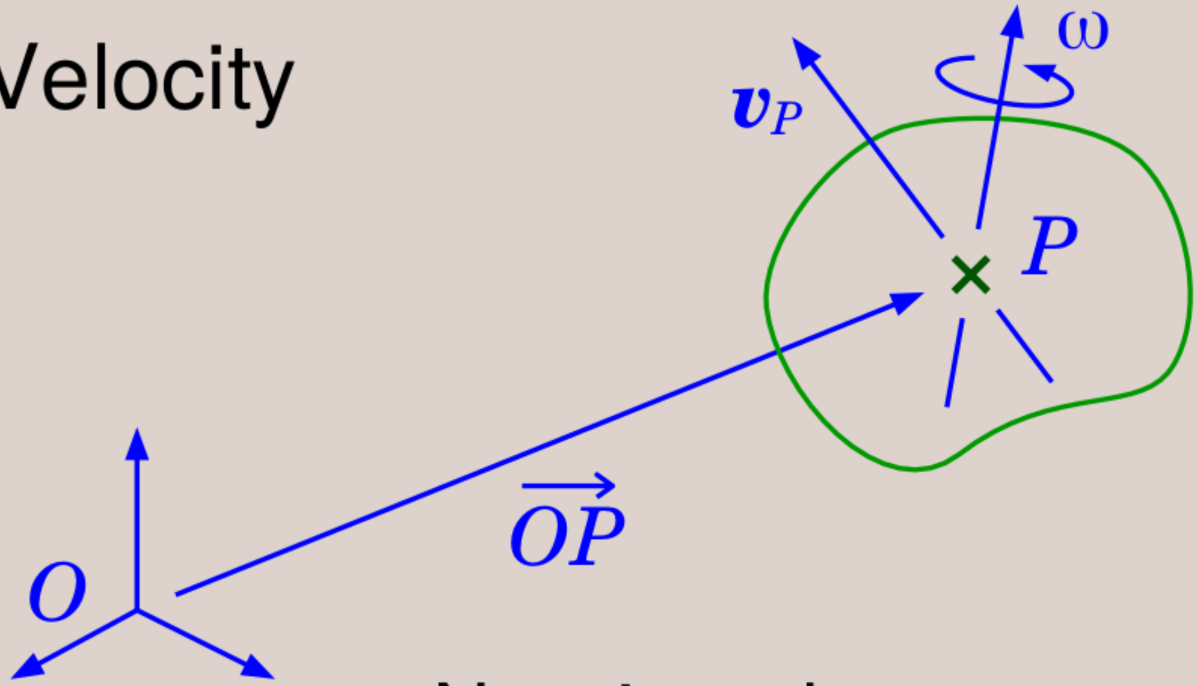
The body is then deemed to be

translating with a linear velocity $\boldsymbol{v}_P$

while simultaneously

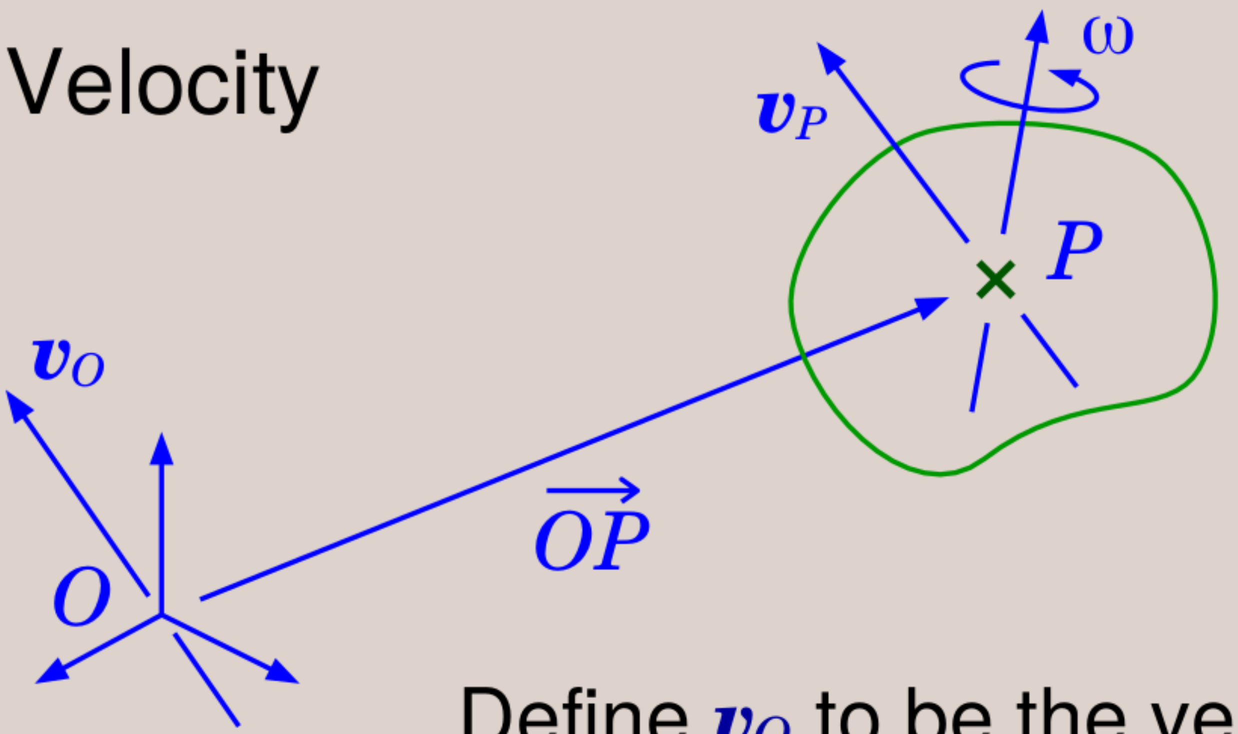
rotating with an angular velocity ω about an axis passing through P

Velocity



Now introduce a coordinate frame with an origin at any fixed point O

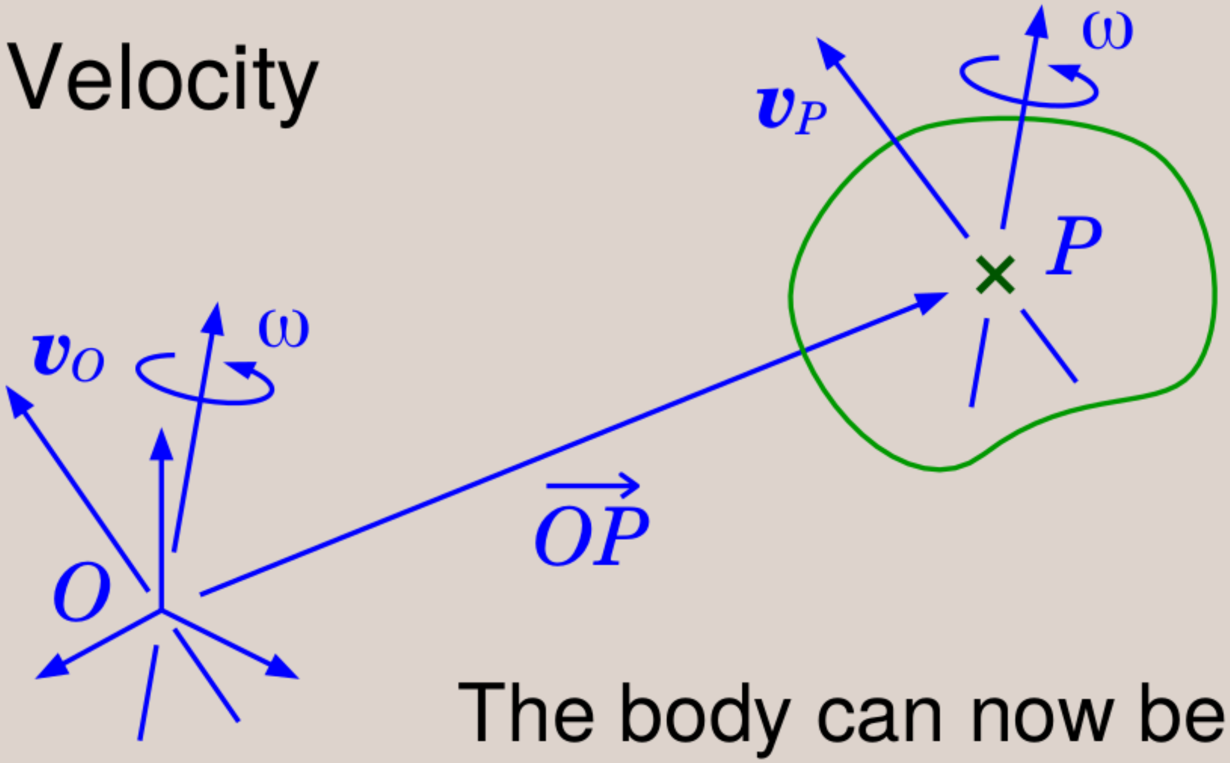
Velocity



Define $\boldsymbol{v}_O$ to be the velocity of the body−fixed point that coincides with O at the current instant

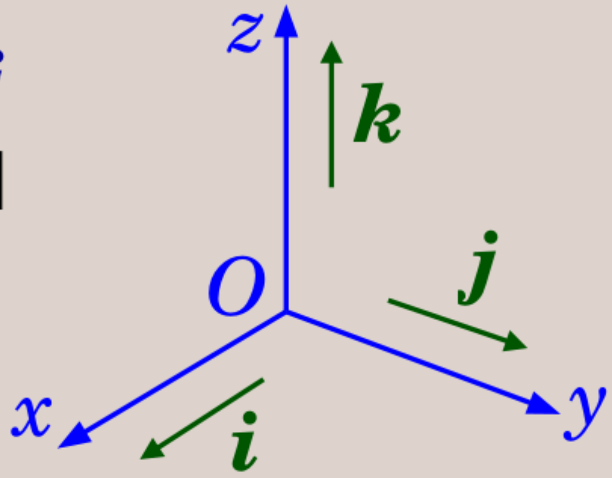
$$\boldsymbol{v}_O = \boldsymbol{v}_P + \overrightarrow{OP} \times \omega$$

Velocity



The body can now be regarded as translating with a velocity of $\boldsymbol{v}_O$ while simultaneously rotating with an angular velocity of ω about an axis passing through O

Introduce the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ pointing in the x, y and z directions.



ω and $\boldsymbol{v}_O$ can now be expressed in terms of their Cartesian coordinates:

$$\underbrace{\underline{\omega} = \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} \quad \underline{v}_O = \begin{bmatrix} v_{Ox}\\ v_{Oy}\\ v_{Oz}\end{bmatrix}}_{\text{coordinate vectors}} \qquad \underbrace{\begin{aligned}\omega &= \omega_x\,\boldsymbol{i} + \omega_y\,\boldsymbol{j} + \omega_z\,\boldsymbol{k}\\ \boldsymbol{v}_O &= v_{Ox}\,\boldsymbol{i} + v_{Oy}\,\boldsymbol{j} + v_{Oz}\,\boldsymbol{k}\end{aligned}}_{\text{what they represent}}$$

The motion of the body can now be expressed as the sum of six elementary motions:

a linear velocity of v_{Ox} in the x direction

\+ a linear velocity of v_{Oy} in the y direction

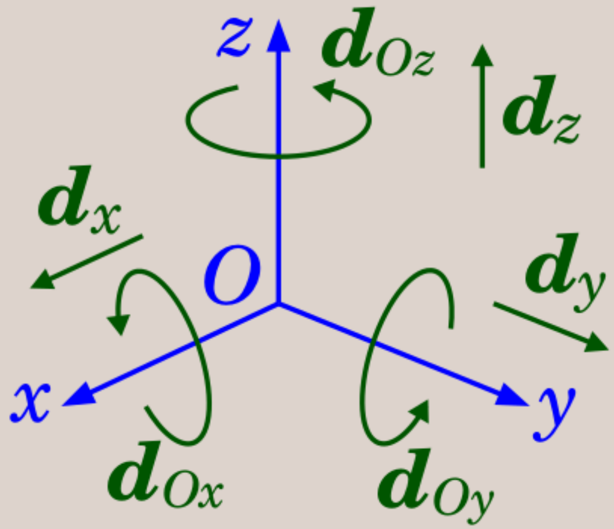
\+ a linear velocity of v_{Oz} in the z direction

\+ an angular velocity of ω_x about the line Ox

\+ an angular velocity of ω_y about the line Oy

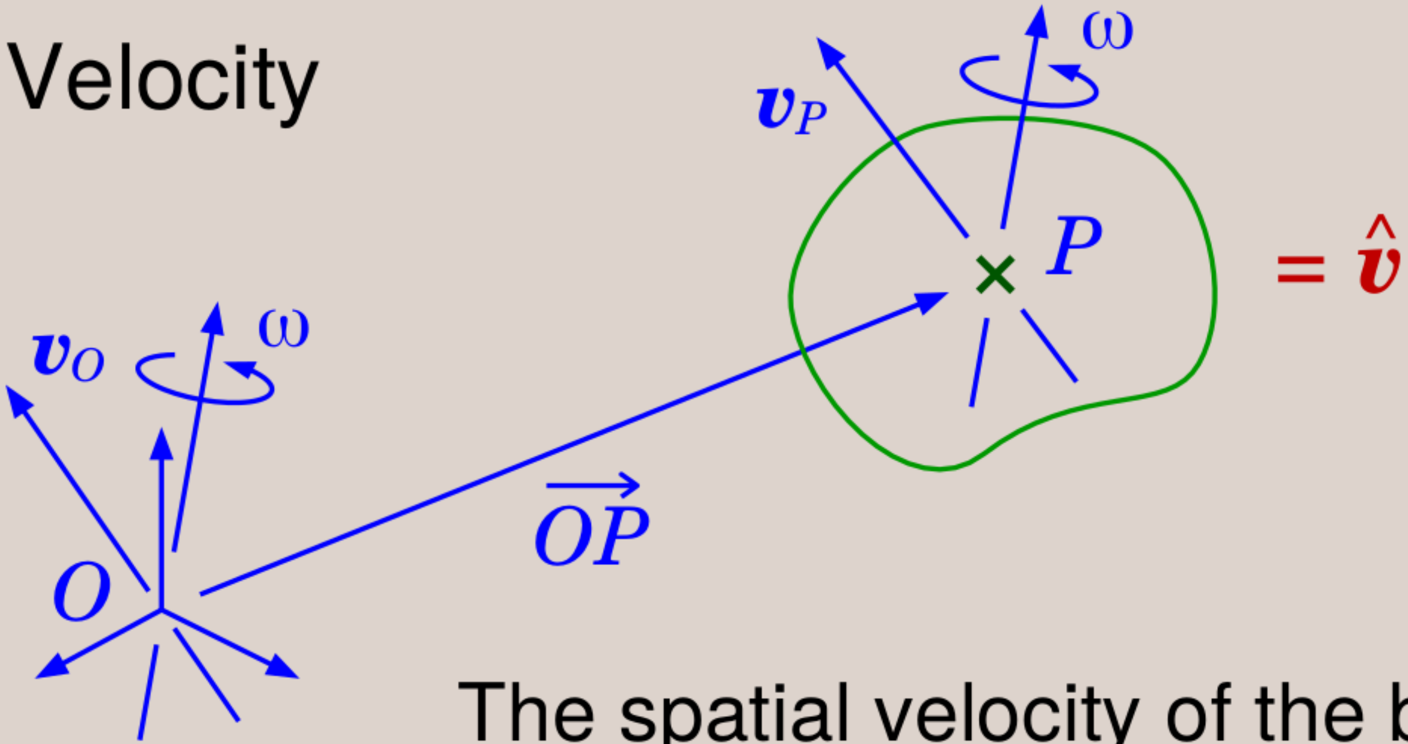
\+ an angular velocity of ω_z about the line Oz

Define the following
Plücker basis on M^6:



$\boldsymbol{d}_{Ox}$ unit angular motion about the line Ox
$\boldsymbol{d}_{Oy}$ unit angular motion about the line Oy
$\boldsymbol{d}_{Oz}$ unit angular motion about the line Oz
$\boldsymbol{d}_x$ unit linear motion in the x direction
$\boldsymbol{d}_y$ unit linear motion in the y direction
$\boldsymbol{d}_z$ unit linear motion in the z direction

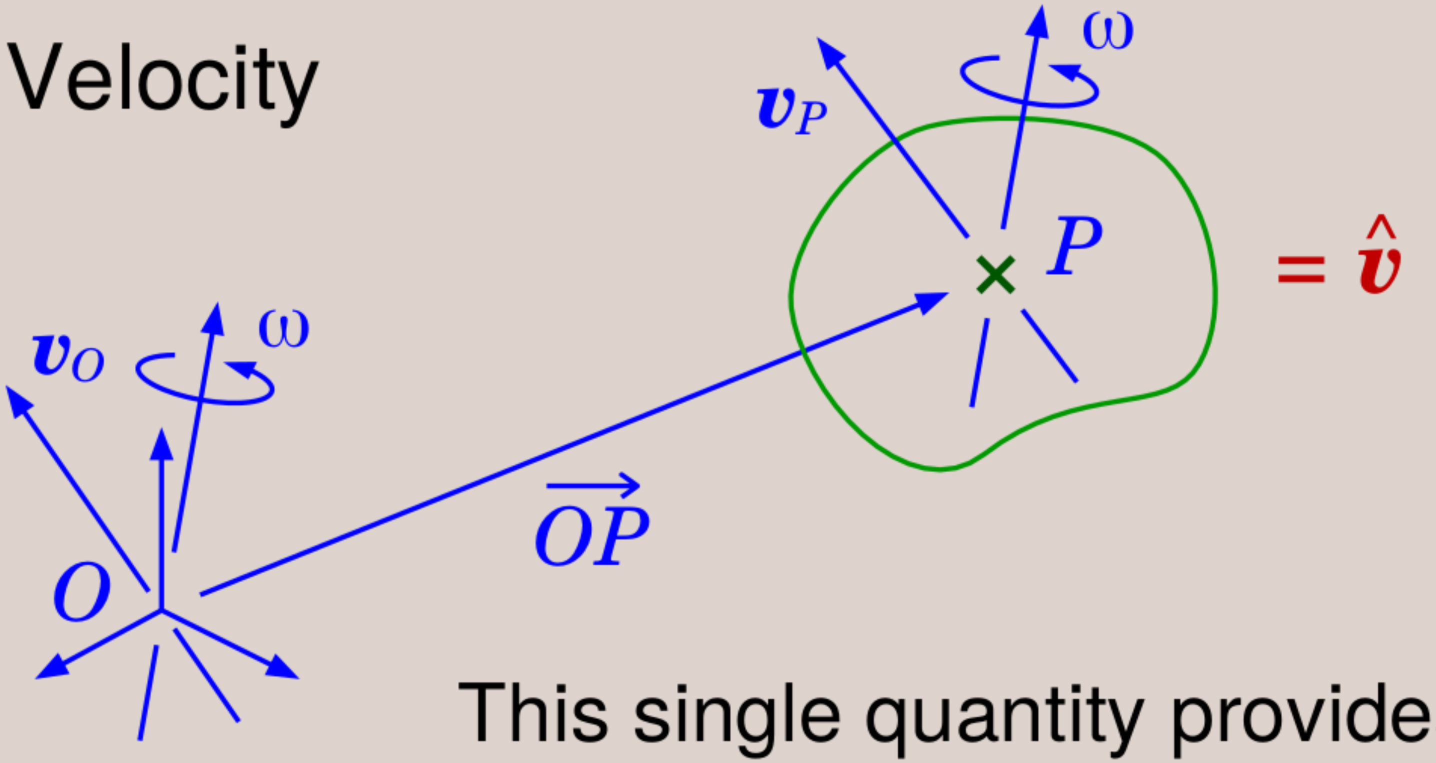
Velocity



The spatial velocity of the body can now be expressed as

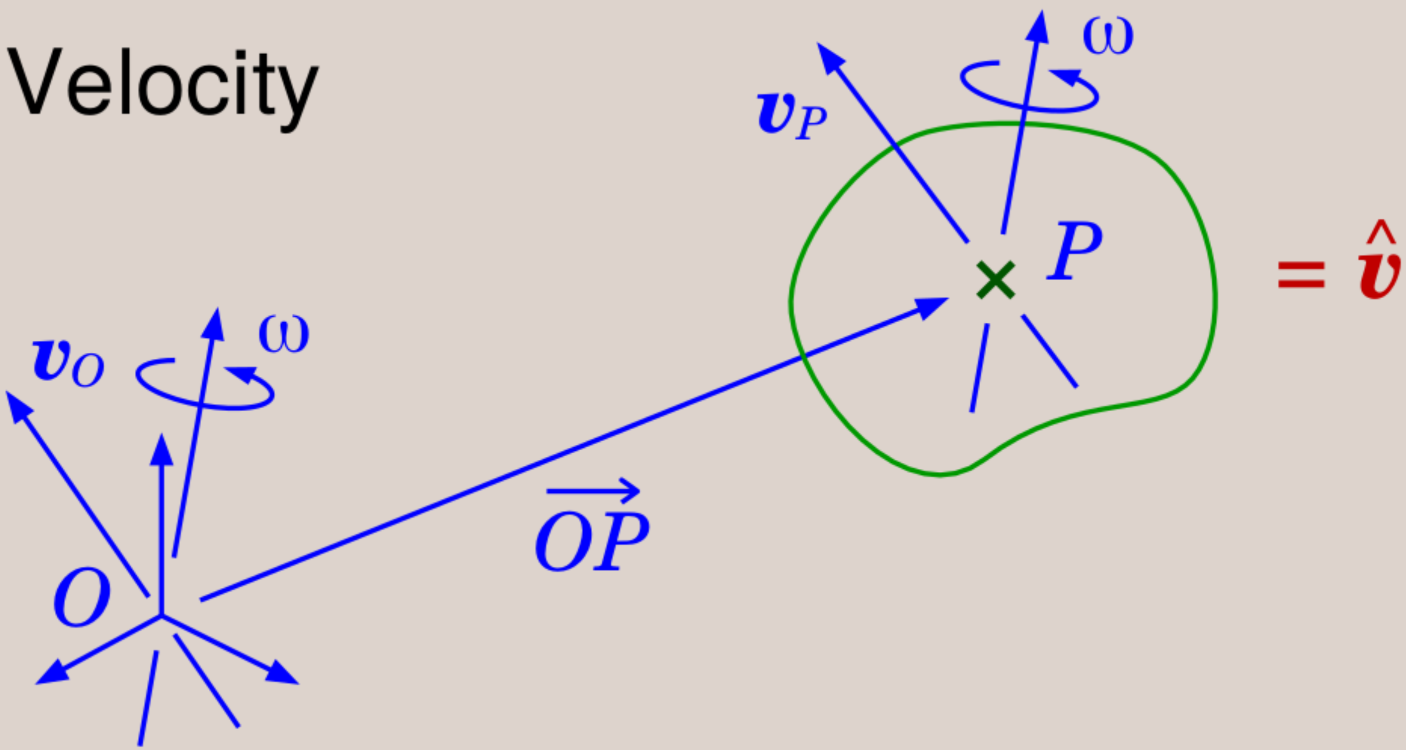
$$\hat{\boldsymbol{v}} = \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + \\ + v_{Ox} \boldsymbol{d}_x + v_{Oy} \boldsymbol{d}_y + v_{Oz} \boldsymbol{d}_z$$

Velocity



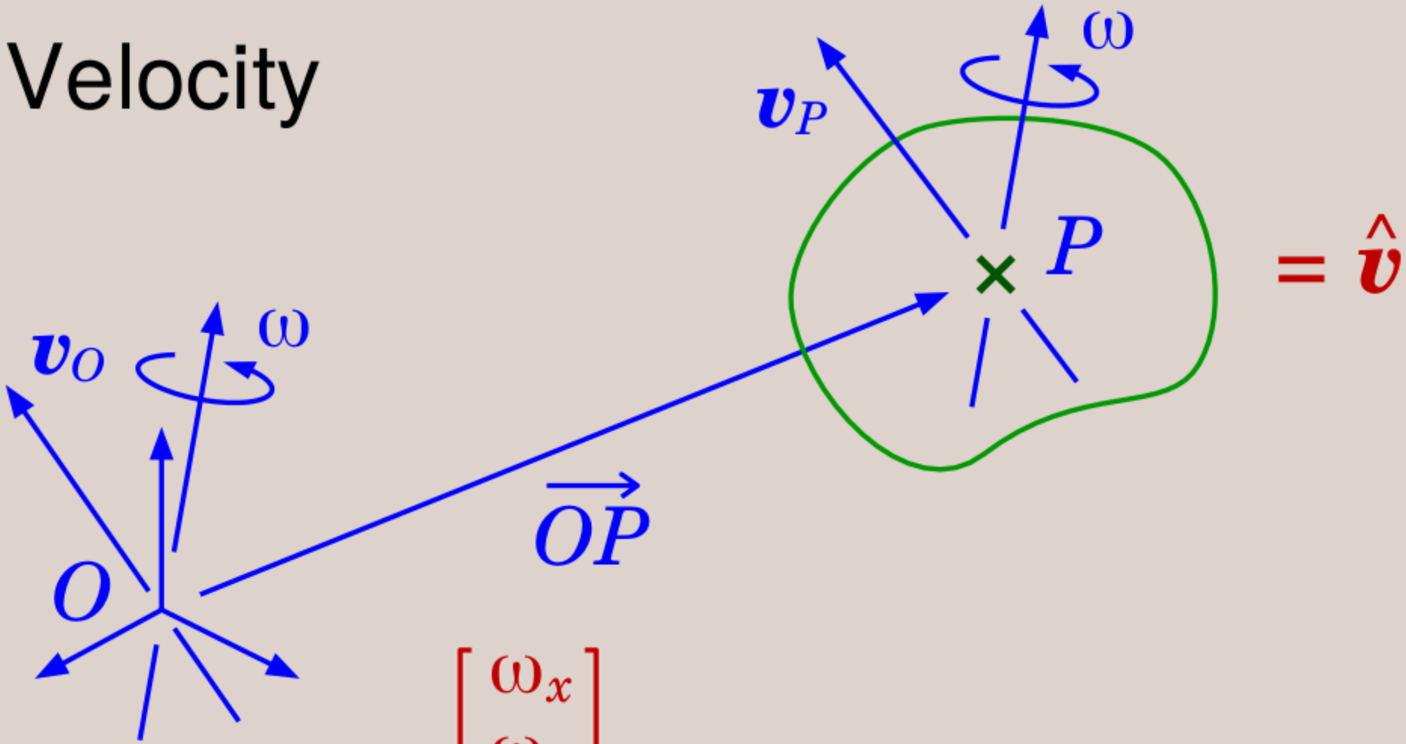
This single quantity provides a *complete description* of the velocity of a rigid body, and it is *invariant* with respect to the location of the coordinate frame

Velocity



The six scalars $\omega_x, \omega_y, \ldots, v_{Oz}$ are the *Plücker coordinates* of $\hat{\boldsymbol{v}}$ in the coordinate system defined by the frame $Oxyz$

Velocity



$$\hat{\underline{\boldsymbol{v}}}_O = \begin{bmatrix} \underline{\omega} \\ \underline{\boldsymbol{v}}_O \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_{Ox} \\ v_{Oy} \\ v_{Oz} \end{bmatrix}$$

coordinate vector

$$\hat{\boldsymbol{v}} = \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + v_{Ox} \boldsymbol{d}_x + v_{Oy} \boldsymbol{d}_y + v_{Oz} \boldsymbol{d}_z$$

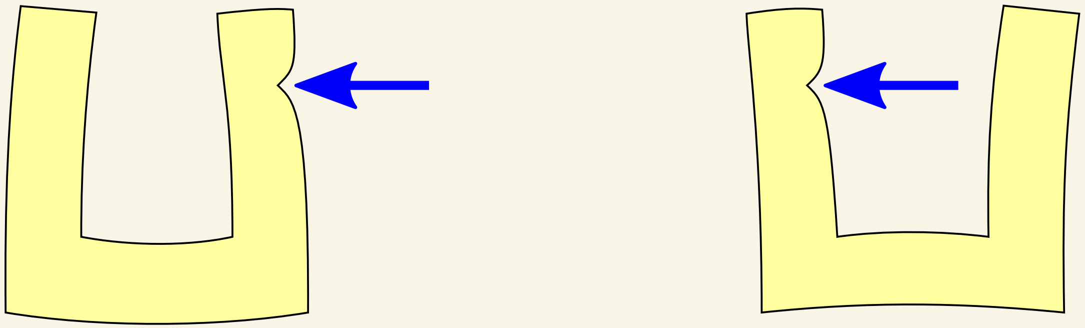
what it represents

Now try question set A in part2Q.pdf

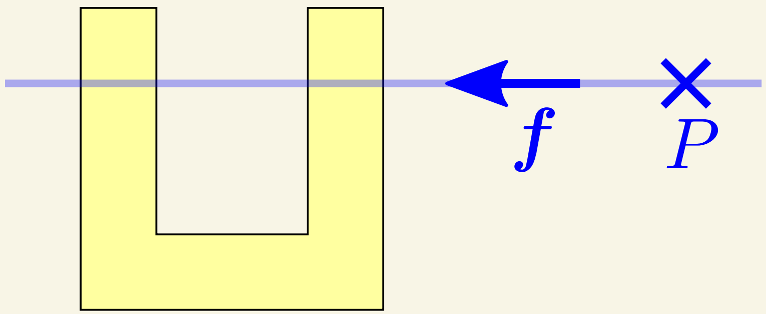
(answers in part2A.pdf)

Linear Forces

When a force acts on a real physical solid, it matters which part of the body feels the force.



But when a force acts on a rigid body, only the magnitude and line of action matter.



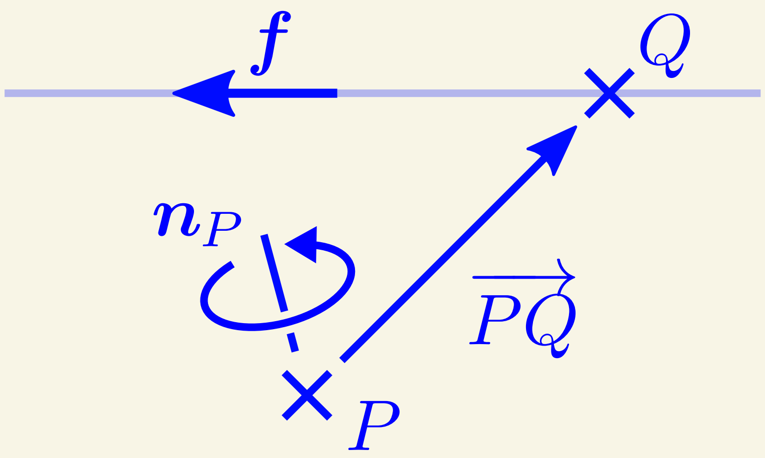
description: Euclidean vector $\boldsymbol{f}$ plus any point P on the line

Angular Forces — Moment, Couple, Torque

- Moment is the turning power of a linear force about a point.
- A force $\boldsymbol{f}$ acting along a line passing through Q produces a moment $\boldsymbol{n}_P$ at point P given by
$$\boldsymbol{n}_P = \overrightarrow{PQ} \times \boldsymbol{f}$$
- The set of all moments produced by a given force forms a *moment vector field*

$$\boldsymbol{N}_Q(P) = \overrightarrow{PQ} \times \boldsymbol{f} = \boldsymbol{n}_P$$

moment field of $\boldsymbol{f}$ through Q



Angular Forces — Moment, Couple, Torque

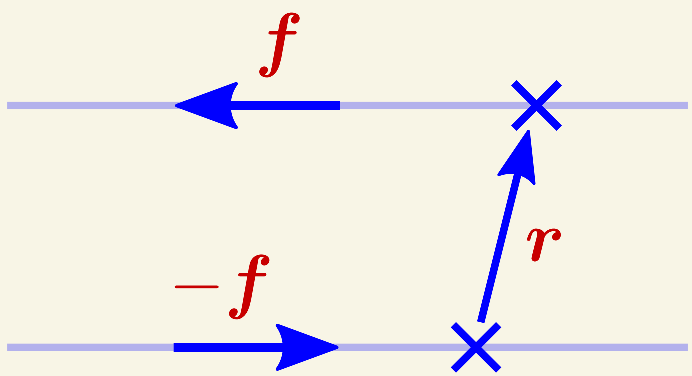
- A force $\boldsymbol{f}$ acting along a line passing through Q is equivalent to a force $\boldsymbol{f}$ acting along a line passing through P plus the moment $\boldsymbol{n}_P$ of the original force about P.

- In classical mechanics (using Euclidean vectors) this result is used to replace a 'system of forces', $\boldsymbol{f}_i$, acting through points Q_i, with a single force $\boldsymbol{f}$ acting through P and a moment $\boldsymbol{n}_P$ where

$$\boldsymbol{f} = \sum_i \boldsymbol{f}_i \qquad \text{and} \qquad \boldsymbol{n}_P = \sum_i \overrightarrow{PQ_i} \times \boldsymbol{f}_i$$

Angular Forces — Moment, Couple, Torque

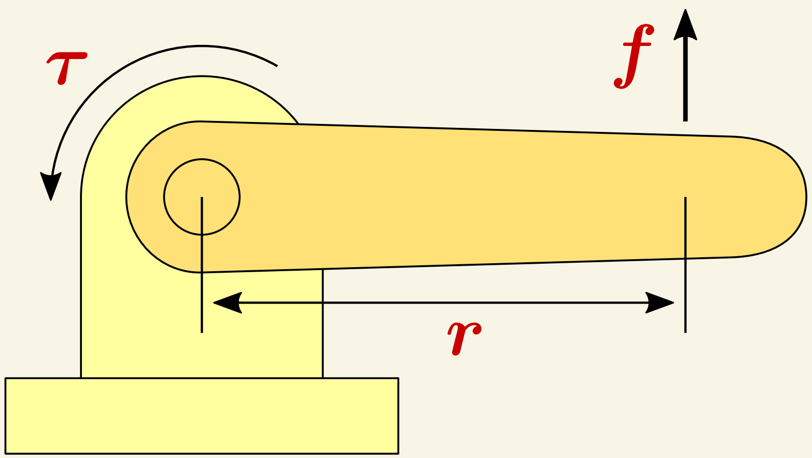
- A couple is the sum of two forces of equal magnitude and opposite direction acting along parallel lines.
- Couple is a vector having magnitude and direction but no line of action.



couple $\boldsymbol{n} = \boldsymbol{r} \times \boldsymbol{f}$

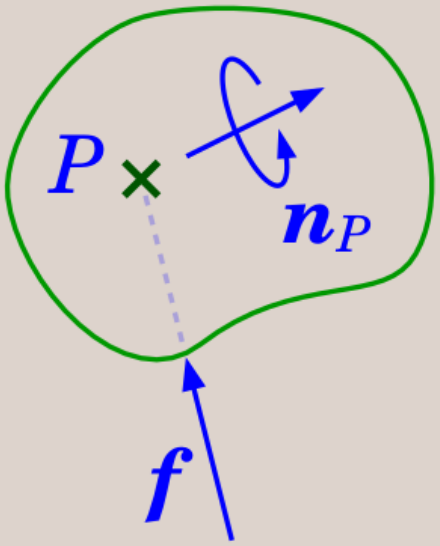
Angular Forces — Moment, Couple, Torque

- Torque is the turning force on a supported shaft.
- Torque is a scalar.
- A torque τ on a shaft spinning at speed ω delivers mechanical power $\tau\omega$.



A torque τ can deliver a force $f = \tau / r$ at a radius r.

Force

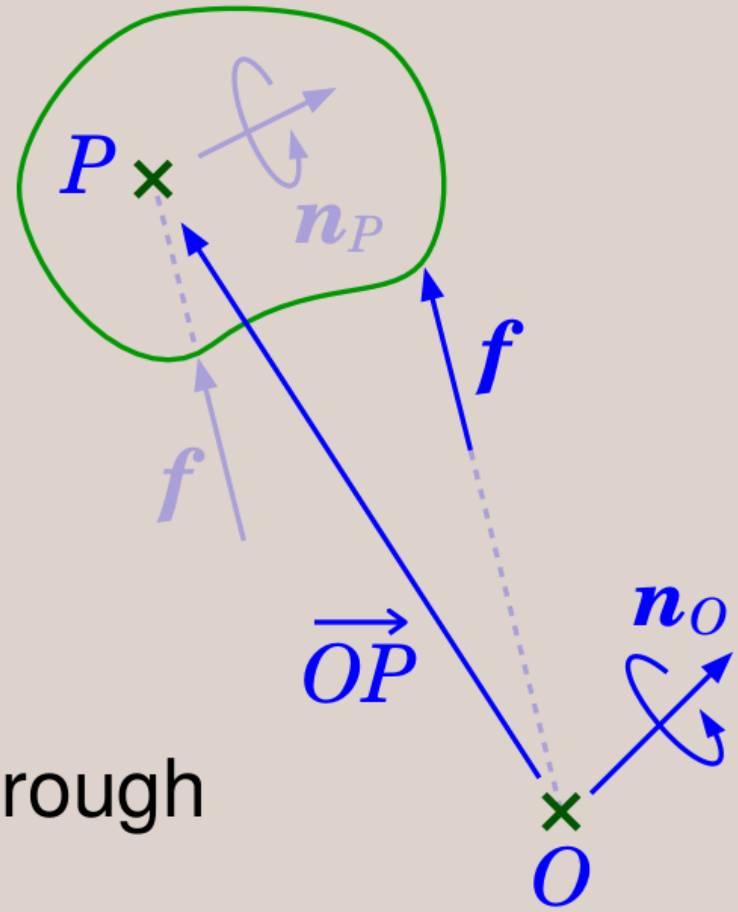


A general force acting on a rigid body can be expressed as the sum of

- a linear force $\boldsymbol{f}$ acting along a line passing through any chosen point P, and
- a couple, $\boldsymbol{n}_P$

Force

If we choose a different point, $\boldsymbol{O}$, then the force can be expressed as the sum of



- a linear force $\boldsymbol{f}$ acting along a line passing through the new point $\boldsymbol{O}$, and
- a couple $\boldsymbol{n}_O$, where $\boldsymbol{n}_O = \boldsymbol{n}_P + \overrightarrow{OP} \times \boldsymbol{f}$

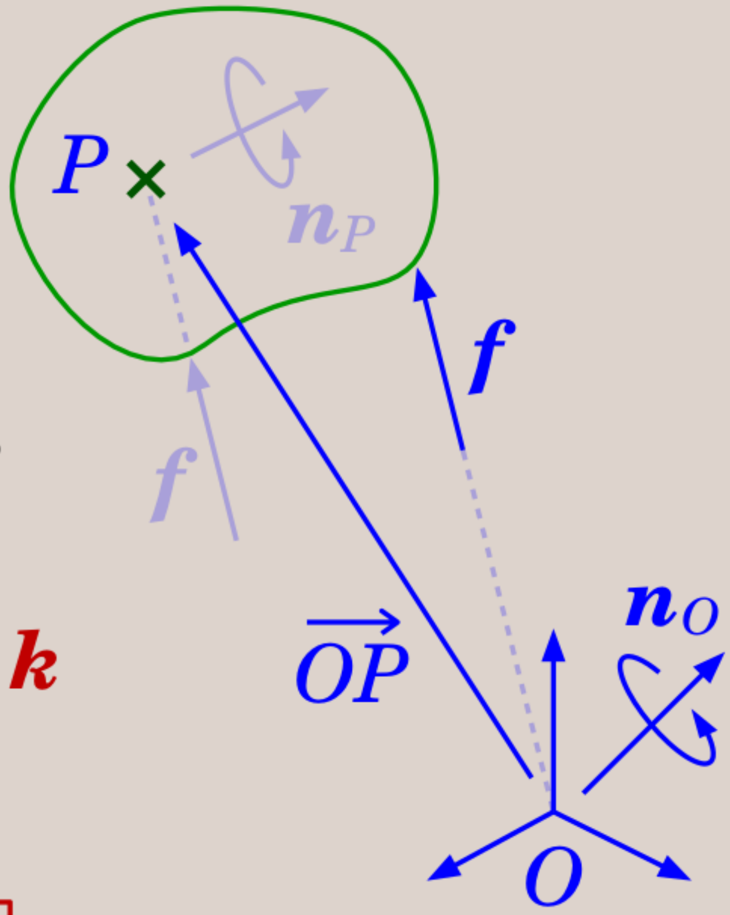
Force

Now place a coordinate frame at O and introduce unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, as before, so that

$$\boldsymbol{n}_O = n_{Ox}\,\boldsymbol{i} + n_{Oy}\,\boldsymbol{j} + n_{Oz}\,\boldsymbol{k}$$

$$\boldsymbol{f} = f_x\,\boldsymbol{i} + f_y\,\boldsymbol{j} + f_z\,\boldsymbol{k}$$

$$\underline{\boldsymbol{n}}_O = \begin{bmatrix} n_{Ox} \\ n_{Oy} \\ n_{Oz} \end{bmatrix} \qquad \underline{\boldsymbol{f}} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$



The total force acting on the body can now be expressed as the sum of six elementary forces:

a moment of n_{Ox} in the x direction

\+ a moment of n_{Oy} in the y direction

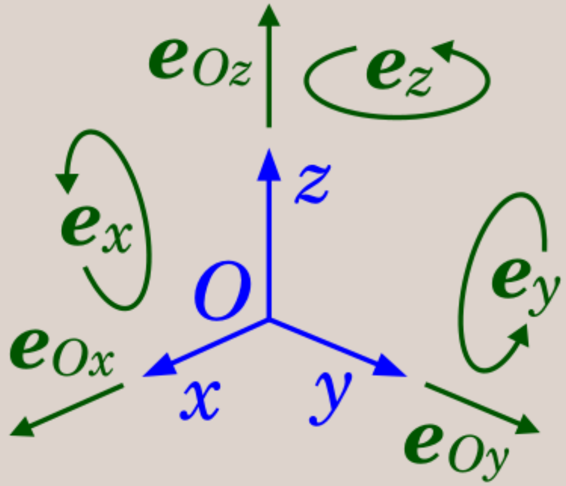
\+ a moment of n_{Oz} in the z direction

\+ a linear force of f_x acting along the line Ox

\+ a linear force of f_y acting along the line Oy

\+ a linear force of f_z acting along the line Oz

Define the following
Plücker basis on F^6:

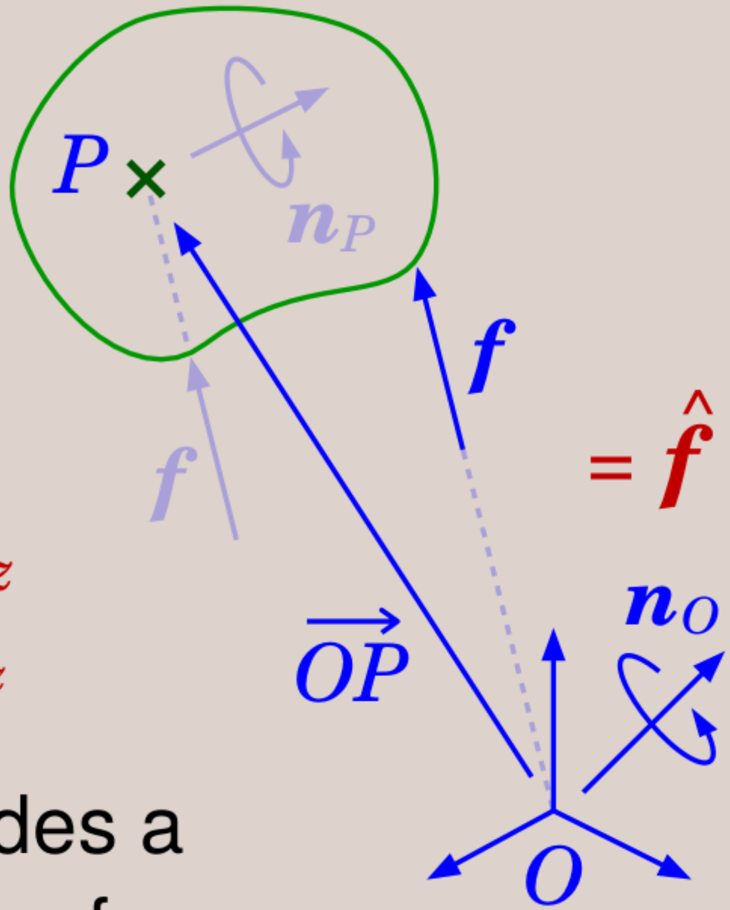


$\boldsymbol{e}_x$	unit couple in the x direction
$\boldsymbol{e}_y$	unit couple in the y direction
$\boldsymbol{e}_z$	unit couple in the z direction
$\boldsymbol{e}_{Ox}$	unit linear force along the line Ox
$\boldsymbol{e}_{Oy}$	unit linear force along the line Oy
$\boldsymbol{e}_{Oz}$	unit linear force along the line Oz

Force

The spatial force acting on the body can now be expressed as

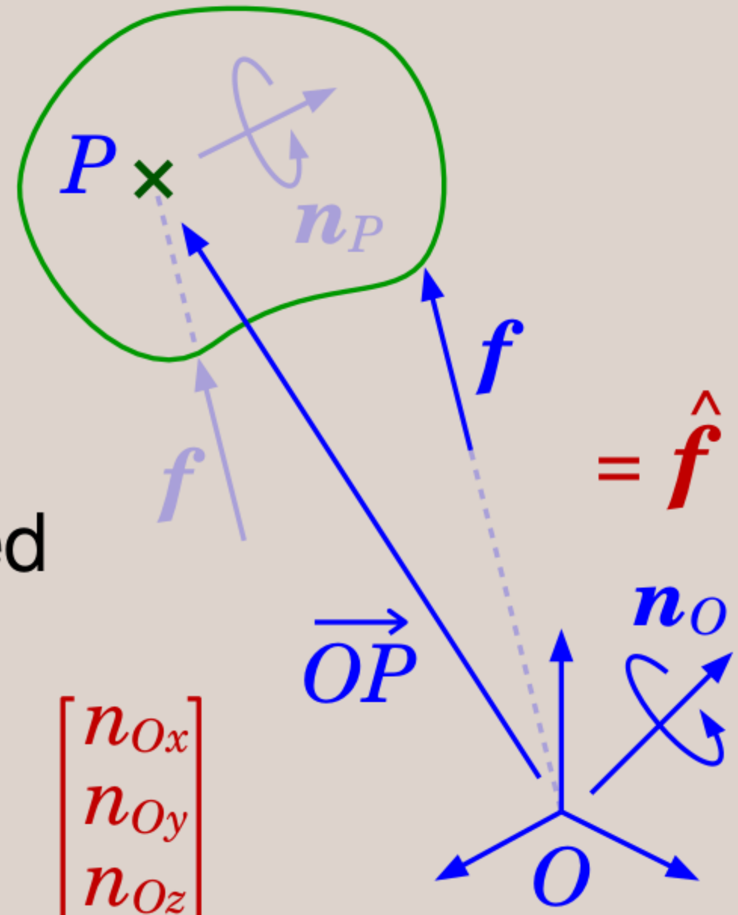
$$\hat{\boldsymbol{f}} = n_{Ox}\,\boldsymbol{e}_x + n_{Oy}\,\boldsymbol{e}_y + n_{Oz}\,\boldsymbol{e}_z + f_x\,\boldsymbol{e}_{Ox} + f_y\,\boldsymbol{e}_{Oy} + f_z\,\boldsymbol{e}_{Oz}$$



This single quantity provides a *complete description* of the forces acting on the body, and it is *invariant* with respect to the location of the coordinate frame

Force

The six scalars n_{Ox}, $n_{Oy}, \ldots, f_z$ are the *Plücker coordinates* of $\hat{\boldsymbol{f}}$ in the coordinate system defined by the frame $Oxyz$



coordinate vector: $\underline{\hat{f}}_O = \begin{bmatrix} \underline{\boldsymbol{n}}_O \\ \underline{\boldsymbol{f}} \end{bmatrix} = \begin{bmatrix} n_{Ox} \\ n_{Oy} \\ n_{Oz} \\ f_x \\ f_y \\ f_z \end{bmatrix}$

Two Vector Spaces

For technical reasons, we cannot place all spatial vectors in a single vector space; so we have to use two:

M^6 for spatial motion vectors (velocity, acceleration, . . .)

F^6 for spatial force vectors (force, momentum, impulse, . . .)

This is why we need two sets of basis vectors: $\boldsymbol{d}_{Ox}, \ldots, \boldsymbol{d}_z$ to span M^6 and $\boldsymbol{e}_x, \ldots, \boldsymbol{e}_{Oz}$ to span F^6.

A Plücker coordinate system uses 12 basis vectors, which span 2 vector spaces, but all 12 are defined using the *same Cartesian frame*.

Plücker Coordinates are Dual

scalar products of basis vectors

The basis vectors satisfy the 'reciprocity' condition, so Plücker coordinates are a system of dual coordinates covering two vector spaces.

$\cdot$	$\boldsymbol{e}_x$	$\boldsymbol{e}_y$	$\boldsymbol{e}_z$	$\boldsymbol{e}_{Ox}$	$\boldsymbol{e}_{Oy}$	$\boldsymbol{e}_{Oz}$
$\boldsymbol{d}_{Ox}$	1	0	0	0	0	0
$\boldsymbol{d}_{Oy}$	0	1	0	0	0	0
$\boldsymbol{d}_{Oz}$	0	0	1	0	0	0
$\boldsymbol{d}_x$	0	0	0	1	0	0
$\boldsymbol{d}_y$	0	0	0	0	1	0
$\boldsymbol{d}_z$	0	0	0	0	0	1

Given coordinate vectors $\underline{\boldsymbol{m}}, \underline{\boldsymbol{f}} \in \mathsf{R}^6$ representing $\boldsymbol{m} \in \mathsf{M}^6$ and $\boldsymbol{f} \in \mathsf{F}^6$ in the same Plücker coordinate system, the scalar product is

$$\boldsymbol{m} \cdot \boldsymbol{f} = \underline{\boldsymbol{m}}^{\mathrm{T}} \underline{\boldsymbol{f}}$$

Basic Operations with Spatial Vectors

- Rigid Connection

  If two bodies are rigidly connected, or are moving in rigid formation, then their velocities are the same.

- Relative Velocity

  If bodies A and B have velocities of $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ then the relative velocity of B with respect to A is

$$\boldsymbol{v}_{\mathrm{rel}} = \boldsymbol{v}_B - \boldsymbol{v}_A$$

- Summation of Forces

  If forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ both act on the same body then they are equivalent to a single force $\boldsymbol{f}_{\mathrm{tot}}$ given by

$$\boldsymbol{f}_{\mathrm{tot}} = \boldsymbol{f}_1 + \boldsymbol{f}_2$$

Basic Operations with Spatial Vectors

- Action and Reaction

  If body A exerts a force $\boldsymbol{f}$ on body B then body B exerts a force $-\boldsymbol{f}$ on body A. (Newton's 3rd law in spatial form.)

- Scalar Product

  If a force $\boldsymbol{f}$ acts on a body with velocity $\boldsymbol{v}$ then the power delivered by that force is

  $$power = \boldsymbol{f} \cdot \boldsymbol{v}$$

  If coordinate vectors $\underline{\boldsymbol{f}}$ and $\underline{\boldsymbol{v}}$ represent $\boldsymbol{f}$ and $\boldsymbol{v}$ in the same dual coordinate system then

  $$power = \underline{\boldsymbol{f}}^{\mathrm{T}} \underline{\boldsymbol{v}}$$

Basic Operations with Spatial Vectors

- Limitation on Scalar Product

  A scalar product is defined *only* between a motion vector and a force vector. Expressions like $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2$ and $\boldsymbol{f}_1 \cdot \boldsymbol{f}_2$ do not make physical sense and are *not* defined.

- Scalar Multiples

  Multiplying a spatial vector by a scalar α scales both its linear and angular magnitudes by the same amount, but does not affect the directed line. If $\boldsymbol{v}$ is a general screwing motion, for example, then $\alpha\boldsymbol{v}$ has the same pitch and screw axis as $\boldsymbol{v}$ but α times the speed. Likewise, $\beta\boldsymbol{f}$ has the same line of action as $\boldsymbol{f}$ but β times the intensity.

Basic Operations with Spatial Vectors

Later on we will encounter several more basic operations, such as

- the rules for coordinate transforms
- summation of inertias
- momentum = inertia x velocity
- the formula for kinetic energy
- the equation of motion

Now try question set B in part2Q.pdf

(answers in part2A.pdf)